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Rigid body motions and Arnol'd's theory of fronts on $\mathbb{S}^2 \subset \mathbb{R}^3$

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Abstract

To any co-oriented curve (front) in the two-dimensional sphere $\mathbb{S}^2$ we associate a rigid body motion together with an instantaneous axis of rotation. We prove that the pair of (antipodal) curves on the sphere determined by the instantaneous axis of rotation coincide with the envelope of the great circles normal to the original co-oriented curve (this envelope is called the *caustic* of the curve). The cusps of the caustic correspond to the points of the co-oriented curve for which the instantaneous axis of rotation is stationary. These results are stated and proved in the setting of Legendrian curves and contact geometry.

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1. Introduction and results

A co-oriented wave front on the sphere $\mathbb{S}^2$ together with the family of all its equidistant fronts can be considered as wave front propagating eternally on the sphere. To this wave front propagation, we associate a rigid body motion whose instantaneous axis of rotation describes a curve on the sphere which coincides with the caustic of the system of propagating fronts.

Here, $\mathbb{R}^3$ denotes the three-dimensional Euclidean space with the standard orientation. Moreover, $\mathbb{S}^2$ denotes the two-dimensional sphere of radius 1 in $\mathbb{R}^3$. Consider a base $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the tangent space of $\mathbb{S}^2$ at a point Q. Let $\mathbf{n}$ be the outward unit normal vector to the sphere at Q. The base $\{\mathbf{e}_1, \mathbf{e}_2\}$ is said to be *positive* if the base $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ of the tangent space of $\mathbb{R}^3$ at Q is positive.

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To formulate our results we start considering a *front* in the two-dimensional sphere $\mathbb{S}^2$ as a co-oriented curve having as possible singularities double points or semi-cubical cusps. The precise definition of front, which includes a more general class of co-oriented curves, will be given in Section 2, in the general setting of contact geometry.

With every point of a front $\gamma : s \mapsto \gamma(s) \in \mathbb{S}^2$ on the two-dimensional sphere $\mathbb{S}^2$ of radius 1 in $\mathbb{R}^3$ we will associate a trihedral. The *co-orienting normal* $\mathbf{c}(s)$ is a unit vector tangent to $\mathbb{S}^2$ and orthogonal to γ at $\gamma(s)$. The *unit tangent* vector $\mathbf{b}$ at a point of γ is a unit vector tangent to γ chosen so that the orientation of the sphere, given by $\mathbf{b}(s)$ and the co-orienting normal $\mathbf{c}(s)$, is positive. The spherical curve γ defines by itself at each of its points a unit vector $\mathbf{a}(s) = \gamma(s)$. So to each point of a co-oriented curve on $\mathbb{S}^2$ we associate the trihedral $\mathbf{a}(s), \mathbf{b}(s), \mathbf{c}(s)$, that we call the *natural trihedral*. The end point of the vector $\mathbf{b}$, describes a curve on the sphere that we call (following [3]) the *derivative curve* of the front.

These definitions apply not only to smoothly immersed curves, but also to wave fronts, having cusps.

With the parameter s considered as the time, the motion of the natural trihedral is a rigid body motion about the origin, this motion has an instantaneous axis of rotation. Of course, the angular velocity ω_1 of the rigid motion around this axis depends on the parameterization of the front. In addition, the instantaneous axis of rotation (which, as we will see, always lies in the plane, spanned by $\mathbf{a}$ and $\mathbf{c}$) has also an instantaneous angular velocity ω_2 around the origin of the plane $(\mathbf{a}, \mathbf{c})$. This angular velocity ω_2 also depends on the parameterization.

It is noteworthy to remark that *the quotient $K_n = \omega_2/\omega_1$ of these two angular velocities does not depend on the parameterization. This quotient is the geodesic curvature of the derivative curve of the front.*

Theorem 1. *If γ is a spherical curve having geodesic curvature κ, then the instantaneous axis of rotation of the natural trihedral $\mathbf{a}, \mathbf{b}, \mathbf{c}$ at the time s is determined by the vector*

$$\mathbf{r} = \frac{\kappa}{\sqrt{1+\kappa^2}}\mathbf{a} + \frac{1}{\sqrt{1+\kappa^2}}\mathbf{c} = \sin\theta \cdot \mathbf{a} + \cos\theta \cdot \mathbf{c},$$

where θ is the angle from $\mathbf{c}$ to $\mathbf{r}$. Moreover, if the parameter s is the arc length, then the angular velocity of the rigid motion is equal to $\omega = \sqrt{1+\kappa^2}$.

The unit vector $\mathbf{r} = \sin\theta \cdot \mathbf{a} + \cos\theta \cdot \mathbf{c}$ and the spherical curve $\mathbf{R}$ described by it are called, respectively, the *rotation vector* and the *rotation indicatrix* of the spherical curve γ. That is, the rotation indicatrix and its antipodal curve are the curves described on the sphere by the instantaneous axis of rotation of the natural trihedral.

Note that when the spherical curve γ has a cusp, i.e. when $\cos\theta = 0$, the rotation vector is well-defined and is equal to $\mathbf{a}$.

Let Γ be a co-oriented front on an oriented sphere $\mathbb{S}^2$ of radius 1. The great circles of the sphere, orthogonal to the front, are called its *rays*. The rays are oriented by the co-orientation of the front. The envelope of the system of rays of a co-oriented front Γ is a curve with two connected components which is called the *evolute* or *caustic* of that front.

Remark. If the front is parameterized then the two components of the caustic are also parameterized. The monodromy is trivial.

On moving each point of the front along the ray by a distance t we obtain a new front Γ_t called the *t-equidistant* of the front.

Theorem 2. *The rotation indicatrix* **R** *of a front and its antipodal curve* $-$**R** *are the two components of the evolute of that front.*

A smooth parameterization of the fronts by arc length is not convenient because such a parameterization is not possible at the cusps. It is known [3,5] that the derivative curve of a front has no cusp. So, given a front Γ, we propose to parameterize its derivative Γ' by arc length. This parameterization of Γ' induces a parameterization of the front Γ, which we call the *induced parameterization*.

Theorem 3. *Given a spherical co-oriented curve* (*front*), *the natural trihedral of all its equidistants, parameterized with the induced parameterization, define the same rigid body motion*. *The angular velocity around the instantaneous axis of rotation is 1*.

The Maslov index μ of an oriented and co-oriented front, defined in [2], is equal to the algebraic number of cusps of the front. A semi-cubical cusp is said to be *positive* if the cords that join points of the branch approaching the cusp to points of the branch leaving it co-orient the front positively in a neighborhood of the cusp.

Definition. A *rotation vertex* of a front on the sphere $\mathbb{S}^2$ is a point where the first derivative of the rotation indicatrix vanishes (i.e. it corresponds to a cusp of the rotation indicatrix if the point is a generic rotation vertex).

Remark. In other words, the rotation vertices of a front are the points for which the instantaneous axis of rotation (of the associated rigid motion) is stationary.

Theorem 4. *The rigid motion associated to a closed co-oriented front with Maslov index* $\mu = 0$ *has at least two rotation vertices*. *For each even number* $\mu \neq 0$, *there exist fronts with Maslov index* μ *and having no rotation vertex* (*see* Fig. 1).

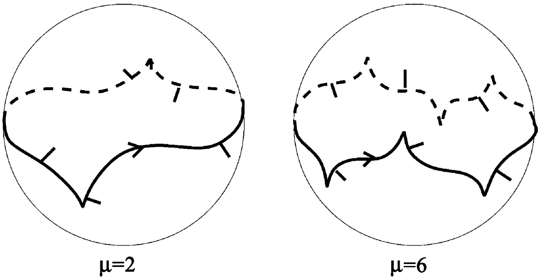

Fig. 1. Two fronts with Maslov index $\mu \not\equiv 0$ and without rotation vertices.

When a point moves along a curve in the Euclidean space $\mathbb{R}^3$, its Frenet trihedral $(\mathbf{t}, \mathbf{n}, \mathbf{b})$, attached to a fixed point, defines a rigid motion whose instantaneous axis of rotation is determined by the Darboux vector: $\tilde{\mathbf{d}} = \tau\mathbf{t} + k\mathbf{b}$, where k and τ are the curvature and the torsion of the curve, respectively. Of course, the angular velocity ω_1 about this axis depends on the parameterization.

Additionally, the instantaneous axis of rotation (which always lies in the rectifying plane, determined by $\mathbf{t}$ and $\mathbf{b}$) has also an instantaneous angular velocity ω_2 around the origin in the rectifying plane. This angular velocity ω_2 also depends on the parameterization.

The end points of the vectors of the Frenet trihedral $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ and of the normalized Darboux vector $\mathbf{d} = \tilde{\mathbf{d}}/\sqrt{k^2 + \tau^2}$ describe four curves $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{D}$, on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, called the *tangent*, *normal*, *binormal* and *Darboux indicatrices* of γ, respectively.

Our results imply that *the quotient* $K_n = \omega_2/\omega_1$ *of the angular velocities* ω_2 *and* ω_1 *does not depend on the parameterization*. *This quotient is the geodesic curvature of the normal indicatrix*.

Generically the Darboux indicatrix may have semi-cubical cusps, corresponding to the points at which the line of the instantaneous axis of rotation is stationary. These points are called *Darboux vertices* [9,10].

A point of a curve γ in $\mathbb{R}^3$, is called *twisting* if the tangent indicatrix of γ, considered a spatial curve, has a flattening (point of torsion 0) at the corresponding point. The geometric meaning of the twistings was studied in [9,10], where it was shown that twistings play an important role in the bifurcations of flattenings of curves and that *the twistings correspond one-to-one to the Darboux vertices*.

To try to answer Aicardi's [1] question about the geometric meaning of the flattenings of the normal indicatrix of a curve $\gamma \subset \mathbb{R}^3$, we apply our results to the case when the front is the tangent indicatrix of the curve γ.

Theorem 5. *The flattenings of the normal indicatrix of a curve in* $\mathbb{R}^3$ *correspond to points where the quotient* $K_n = \omega_2/\omega_1$ *of the angular velocities* ω_1 *and* ω_2 *of this rigid motion is critical*.

Remark. Other evident interpretation: the flattenings of the normal indicatrix correspond to the cusps of the second caustic, i.e. the caustic of the caustic.

In Section 2, we recall some facts of contact geometry. Then, in Section 3, our results are reformulated and proved in the general setting of contact geometry, of the Legendrian curves and their fronts.

2. The manifold of contact elements, Legendrian submanifolds and their fronts

We recall from [3] some basics of the general theory of contact geometry, Legendrian submanifolds and their fronts. Here, we will only consider a particular class of contact manifolds which will be defined below: the manifold of co-oriented contact elements of a given submanifold B^n.

A linear hyperplane $\mathbb{R}^{n-1}$ of a tangent space to an n-dimensional smooth manifold B^n is said to be a *contact element* of B^n. A co-orientation of a contact element is the choice of one of the two halves into which it divides the tangent space. A co-orientation of a contact element of a Riemannian manifold is determined by a choice of a direction on the line normal to it.

The set of all co-oriented contact elements on a given n-dimensional manifold B^n is fibered over B^n, and the fiber over a point of B^n is the spherized cotangent space of B^n at that point, called the *point of contact*. Thus the set of all the co-oriented contact elements of B^n form the bundle space E^{2n-1} of a smooth fibration

$$\pi : ST^*B^n \to B^n$$

with fiber $\mathbb{S}^{n-1}$. It is called the *spherized cotangent bundle* ST^*B^n of B^n.

This bundle E^{2n-1} is equipped with a natural 'tautological' field of tangent hyperplanes. The hyperplane of the tautological field at a point p of E^{2n-1} is the inverse image by π_* of the hyperplane in the space tangent to B^n, which is represented by the point p of the manifold E^{2n-1}. The tautological field of hyperplanes on E^{2n-1} is called the *natural contact structure* of the manifold of contact elements on B^n.

Remark. This natural contact structure can also be defined by the following *skating* rule. The velocity vector of a motion of a contact element belongs to the hyperplane of the contact structure if the velocity of the point of contact belongs to the contact element.

Example (see [3]). The manifold of co-oriented contact elements of the two-dimensional sphere $\mathbb{S}^2$ is the projective space $\mathbb{R}P^3 = \mathbb{S}^3/\pm 1$. The natural contact structure of $ST^*\mathbb{S}^2 \simeq \mathbb{S}^3/\pm 1$ is obtained in [3] from the field of planes orthogonal to the fibers of the Hopf bundle $\mathbb{S}^3 \to \mathbb{S}^2$, under the projection $\mathbb{S}^3 \to \mathbb{S}^3/\pm 1$. More precisely, given an imaginary quaternion u of length 1, multiplication of all the quaternions by u on the *right* supply a complex structure, which depends on u, to the space of quaternions $\mathbb{R}^4$. For each quaternion q the operator of multiplication on the *left* by q in the space of quaternions $\mathbb{R}^4$ is u-complex. Consider the Hopf bundle corresponding to an imaginary quaternion u of length 1

$$\pi_u : \mathbb{S}^3 \to \mathbb{S}^2_u \simeq \mathbb{C}P^1$$

associating with each non-null point of the u-complex line passing through 0, the direction of that line. The fiber of this principal $\mathbb{S}^1$-bundle is the circle $\{z \cdot \mathrm{e}^{ut} : t \in \mathbb{R}/(2\pi\mathbb{Z})\}$, where $z \in \mathbb{S}^3$ is a point of the fiber. On the sphere $\mathbb{S}^3$ of quaternions of length 1, there is a field $\mathbf{u}$, tangent to the fibers of the bundle π_u

$$\mathbf{u}(z) = z \cdot u = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bigg|_{t=0} (z \cdot \mathrm{e}^{ut}).$$

The field of planes on $\mathbb{S}^3$ orthogonal to the vectors of the field $\mathbf{u}$ provides $\mathbb{S}^3$ with a contact structure called the *u-structure*. A submanifold of a contact manifold E^{2n-1} is said to be *integral* if its tangent space at every point belongs to the contact hyperplane.

Definition. A *Legendre submanifold* of a contact manifold is an integral submanifold of maximal dimension: equal to $n - 1$ for a $(2n - 1)$-dimensional manifold. In particular, a

Legendrian curve of a three-dimensional manifold E^3 equipped with a contact structure is an immersed curve in E^3, whose tangent at each point lies in the plane of the contact structure.

Example. Each co-oriented curve γ immersed in a surface B^2 determines a Legendrian curve in the manifold $E^3 = ST^*B^2$ of co-oriented contact elements of B^2. This Legendrian curve consists of the corresponding co-oriented contact elements on B^2, tangent to γ. A point of B^2 also determines a Legendrian curve in E^3. This curve consists of the co-oriented contact elements of B^2 applied at the point (i.e., it is a fiber of the spherized cotangent bundle).

Definition. The *front* of a Legendrian curve $L : \mathbb{S}^1 \to E^3$ of the manifold of co-oriented contact elements of a surface B^2 is the projection $\pi \circ L(\mathbb{S}^1) \subset B^2$ of this Legendrian curve to the surface B^2.

The co-oriented contact elements forming the original Legendrian curve determine a co-orientation of that front. In the sequel, we will consider the manifold $ST^*\mathbb{S}^2$ of co-oriented contact elements of the two-dimensional sphere, its Legendrian curves and their fronts.

In Section 1, we associated a trihedral to each point of a co-oriented curve. After the introduction of the manifold $ST^*\mathbb{S}^2$, we will associate to each co-oriented contact element of the oriented unit sphere its *natural trihedral* **a**, **b**, **c** in the following way: The point of contact on $\mathbb{S}^2$ defines a unit vector **a**; we denote by **c** the co-orienting unit normal vector; the unit vector **b** tangent to the contact element is chosen so that the pair **b**, **c** orients positively the sphere.

Thus, we associate a *natural trihedral* to a front on the sphere by choosing the natural trihedrals of the co-oriented contact elements tangent to the front. In fact, we are associating a trihedral to each point of the corresponding Legendrian curve.

3. The dual, the derivative and the caustic of a spherical front

With every co-oriented curve on the two-dimensional sphere $\mathbb{S}^2$ of radius 1 Arnol'd [3] associated three other curves:

Definition (Arnol'd [3]). The curve *dual* to a given co-oriented curve on the sphere is the curve obtained from the original curve by moving a distance $\pi/2$ along the normals on the side determined by the co-orientation.

This definition applies not only to smoothly immersed curves, but also to wave fronts, having cusps (of semi-cubical type or, in general, of type $x^a = y^{a+1}$).

The dual curve itself is naturally co-oriented and is a wave front equidistant from the original one (lying at a distance $\pi/2$ from it).

The cusps of the original front correspond to points of spherical inflection on the dual front, while the points of spherical inflection on the original one correspond to cusps on the dual. The second dual of a front is antipodal to the original one, while the fourth dual coincides with the original one.

Definition (Arnol'd [3]). The *derivative* of a co-oriented curve on the oriented standard sphere $\mathbb{S}^2$ is the curve obtained by moving each point a distance $\pi/2$ along the great circle tangent to the original curve at that point. The direction of motion along the tangent is chosen so that the orientation of the sphere, given by the direction of the co-orienting normal and the direction of the tangent, is positive.

This definition applies not only to smoothly immersed curves, but also to wave fronts.

The end points of the unit vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ of the natural trihedral associated to the co-oriented contact elements of a front, describe three curves Γ, Γ' and Γ^*, respectively, on the sphere. Of course, the curve Γ coincides with the original front. Comparing the definitions of the dual and the derivative of a co-oriented front, it is evident that the curve Γ' is the derivative of the front and the curve Γ^* is the dual of the front Γ. In particular, Γ^* is an equidistant of Γ.

Theorem A (Arnol'd [3]). *The derivative of a wave front coincides with the derivative of any of its equidistants and is a smoothly immersed curve on* $\mathbb{S}^2$ *even if the original wave front has generic singularities*.

The derivative of a closed wave front is not an arbitrary immersed curve, but it satisfies a topological condition of quantization. In particular, the results of [3] imply the following theorem of Jacobi [6] (see also [4]):

Jacobi's theorem. *If the derivative of a smoothly immersed closed curve has no point of self-intersection, then it divides the sphere into two parts of equal area*.

Remark. A parameterization of a Legendrian curve determines a parameterization of its front and also determines a parameterization of its natural trihedral (even when the front degenerates into a point). By Theorem A, the derivative of a family of equidistant fronts is smooth and can be parameterized by arc length. This parameterization of the derivative induces a parameterization on each Legendrian curve of the family and hence on its natural trihedral. Such parameterization on the Legendrian curve, on its front and on its natural trihedral, will be called the *induced parameterization*. The parameter will be called the *natural parameter*.

Let σ be the arc length parameter of the derivative Γ' of a front Γ on the sphere. With the induced parameterization the natural equations (NEs) of the front (or of its natural trihedral) become

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\sigma} = \cos\theta \cdot \mathbf{b}, \qquad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\sigma} = -\cos\theta \cdot \mathbf{a} + \sin\theta \cdot \mathbf{c}, \qquad \frac{\mathrm{d}\mathbf{c}}{\mathrm{d}\sigma} = -\sin\theta \cdot \mathbf{b},$$

and the rotation vector is

$$\mathbf{r} = \sin\theta \cdot \mathbf{a} + \cos\theta \cdot \mathbf{c},$$

where $\theta = \theta(\sigma)$ is the angle measured from $\mathbf{c}$ to $\mathbf{r}$. Note that these NEs are well-defined even when the front degenerates into a point.

Remark (Arnol'd [3]). An equidistant is orthogonal to the rays emanating from the original front. An equidistant co-oriented by these rays is the front of a Legendrian curve, diffeomorphic and contactomorphic to the original one.

Two Legendrian curves are called *equidistant* if their fronts are equidistant. We reformulate Theorem 3 of Section 1 into a more general form:

Theorem 3′. *Let L_t be a family of equidistant Legendrian curves. The rigid motion associated to L_t with the induced parameterization does not depend on the value of t and has angular velocity 1.*

A *singular point* of a front is a critical value of the projection of the corresponding Legendrian curve. The singular points of generic fronts are semi-cubical cusps.

Consider the family of equidistants of a given co-oriented front on the standard sphere. Even if the original front is not singular, some of its equidistants will have singular points.

Definition (Arnol'd [3]). The *caustic* of a family of fronts equidistant from one another is the curve formed by their singular points.

Remark. Any smooth curve in the space of trihedrals defines a rigid motion together with its associated instantaneous axis of rotation. The Legendrian curves of $ST^*\mathbb{S}^2$ define special curves in the space of trihedrals. Theorems 2 and 3′ assert that given a parameterized Legendrian curve, all its "equidistant" Legendrian curves define the same rigid body motion and the rotation indicatrix of this motion coincides with the caustic of the system of "equidistant" Legendrian curves.

Remark. The caustic of a system of equidistant fronts has other characterizations: (1) It is formed by their centers of curvature; (2) It is the envelope of their rays (in fact it suffices to start with a front of the system to get all the rays); (3) If the front is considered as a spatial curve, then the caustic of the previous definition coincides with the caustic of the (Lagrangian) Gauß map associated to the curve.

Theorem B (Arnol'd [3]). *The derivative of a front is dual to its caustic.*

4. The Proofs

Proof of Theorem 1. We will first obtain the NEs of a smooth spherical curve. At the points in which $\gamma \subset \mathbb{S}^2$ is smooth we can parameterize it locally by arc length l so that $\mathrm{d}\gamma/\mathrm{d}l = \mathbf{b}$, i.e. $\mathbf{a}' = \mathbf{b}$, where derivation with respect to l is denoted by a prime, $\mathrm{d}/\mathrm{d}l = {}'$. With these conditions we have $\langle \mathbf{b}, \mathbf{b}' \rangle = 0$. Thus $\mathbf{a}'' = \mathbf{b}' = \alpha\mathbf{a} + \beta\mathbf{c}$, where α and β are two smooth functions of the parameter l. The geodesic curvature κ of γ is the component of $\mathbf{a}''$ orthogonal to $\mathbf{a}$. Thus $\beta = \kappa$.

By definition, we always have $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Thus

$$\mathbf{c}' = \mathbf{a} \times \mathbf{b}' + \mathbf{a}' \times \mathbf{b} = \mathbf{a} \times (\alpha\mathbf{a} + \kappa\mathbf{c}) = -\kappa\mathbf{b}.$$

Using $\mathbf{b} = \mathbf{c} \times \mathbf{a}$, we obtain in a similar way that $\mathbf{b}' = -\mathbf{a} + \kappa\mathbf{c}$. Thus the NEs of a smoothly immersed spherical curve parameterized by arc length and with geodesic curvature κ are

$$\mathbf{a}' = \mathbf{b}, \qquad \mathbf{b}' = -\mathbf{a} + \kappa\mathbf{c}, \qquad \mathbf{c}' = -\kappa\mathbf{b}.$$

The instantaneous axis of rotation of the natural trihedron $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ must be orthogonal to the vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$. It is evident that the unit vector

$$\mathbf{r} = \frac{\kappa}{\sqrt{1+\kappa^2}}\mathbf{a} + \frac{1}{\sqrt{1+\kappa^2}}\mathbf{c}$$

satisfies this condition.

The vector $\mathbf{b}$ is orthogonal to the instantaneous axis of rotation and is unitary. So the norm of the vector $\mathbf{b}' = -\mathbf{a} + \kappa\mathbf{c}$ gives the angular velocity. That is, the angular velocity is $\omega = \sqrt{1+\kappa^2}$. □

Proof of Theorem 2. For each contact element of a front Γ, the oriented great circle normal to it (i.e. its ray) is contained in the plane normal to the vector $\mathbf{b}$. So the ray is contained in the plane spanned by $\mathbf{c}$ and $\mathbf{a}$. The rotation vector $\mathbf{r} = \sin\theta \cdot \mathbf{a} + \cos\theta \cdot \mathbf{c}$, also lies in this plane. Moreover, the derivative of the rotation vector with respect to the parameter σ is

$$\mathbf{r}' = \theta' \cos\theta \cdot \mathbf{a} - \theta' \sin\theta \cdot \mathbf{c} = -\theta' \cdot \mathbf{b}',$$

which also lies in the plane $(\mathbf{c}, \mathbf{a})$. Thus the ray of the front is tangent to the rotation indicatrix $\mathbf{R}$. This means that the rotation indicatrix $\mathbf{R}$ and its antipodal curve $-\mathbf{R}$ form the evolute of the front and, by Theorem A, they also form the evolute of all fronts equidistant to Γ. □

Proof of Theorem 3′. First, all fronts of the family of equidistant Legendrian curves have the same caustic, i.e. the same rotation indicatrix. This means that the rigid motions associated to this fronts have the same instantaneous axis of rotation. The vector $\mathbf{b}$ is unitary and is orthogonal to the instantaneous axis of rotation. So, the derivative of $\mathbf{b}$ with respect to the natural parameter σ is the angular velocity of the rigid motion. From the NEs corresponding to the induced parameterization we have that

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\sigma} = -\cos\theta \cdot \mathbf{a} + \sin\theta \cdot \mathbf{c}.$$

So, the angular velocity is $(\cos^2\theta + \sin^2\theta)^{1/2} = 1$. □

Proof of Theorem 4. For the points at which the front is smooth, i.e. the front is locally the tangent indicatrix of some curve in the Euclidean 3-space. Therefore, at the rotation vertices of the front the radius of the osculating circle is critical, or equivalently the front—considered as a space curve—has a flattening.

A front having Maslov index $\mu = 0$ has the same number of positive and negative cusps. If this number is 0, then the front is a smooth immersed curve in the sphere. Thus it has at least two flattenings, i.e. it has at least two rotation vertices. If the number of cusps is not 0, then there are at least two arcs of the front joining consecutive cusps of different sign. The following lemma is easy to prove.

Lemma 1. *Between two consecutive cusps of different sign of an oriented and co-oriented front in the sphere there is an odd number* (*at least* 1) *of points at which the radius of the osculating circle is critical*.

By Lemma 1, at each one of these arcs there is an odd number of rotation vertices.

Another proof. The *oriented length* of a generic caustic is the alternating sum of the lengths of its segments between successive cusps.

Theorem C (Arnol'd [3]). *The oriented length of the caustic of a system of closed equidistant fronts on the standard Riemannian two-dimensional sphere is equal to* $\mu\pi$, *where* μ *is the Maslov index of any front of the family*.

Theorem C implies that for $\mu = 0$ the caustic has oriented length equal to 0. If the caustic is not a point, then it has at least two cusps, i.e. it has at least two rotation vertices. If the caustic is a point, then all points of the front are rotation vertices.

To construct a front with Maslov index $\mu = 2k > 0$ and having no rotation vertex it suffices to design a front with exactly $2k$ cusps all of them positives and such that the geodesic curvature between any two consecutive cusps is monotone. To satisfy this condition, the front must have only one spherical inflection between each pair of consecutive cusps. In Fig. 1 we have examples of such kind of fronts for $\mu = 2$ and 6. In [3], Arnol'd gives a way to construct fronts with Maslov index $\mu = 2k$ whose caustic is a parallel of latitude of the sphere of radius 1 traversed p times (at a distance $\theta < \pi/2$ from the North pole along meridians).

5. Remarks on perestroikas of Legendrian curves

Orthonormal-oriented frames in $\mathbb{R}^3$ (trihedrals) form the contact three-dimensional manifold $ST^*\mathbb{S}^2$ of co-oriented contact elements on the sphere. The tangent vector to a curve in this space of frames belongs to the contact plane if the derivative of the first vector (or the derivative of the third vector) of the frame is linearly dependent on the second vector.

The curve swept by the Frenet frame associated with a space curve is Legendrian. The fronts Γ and $\Gamma^\vee$ of this Legendre curve under two natural Legendre projections given by the first and the third vectors of the frame are the tangent and the binormal indicatrices, respectively. The tangent and the binormal indicatrices are equidistant fronts (one of each other) on the sphere. (It is also possible to fix the standard contact structure in $ST^*\mathbb{S}^2$, and to consider the Legendrian curve formed by the contact elements tangent to the tangent indicatrix co-oriented by the binormal vector. Using the results of [3], one proves that there is a Legendrian projection of this Legendrian curve onto the binormal indicatrix).

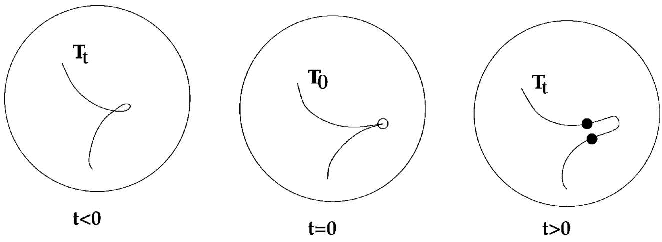

Fig. 2. Cusp transition of the tangent indicatrix.

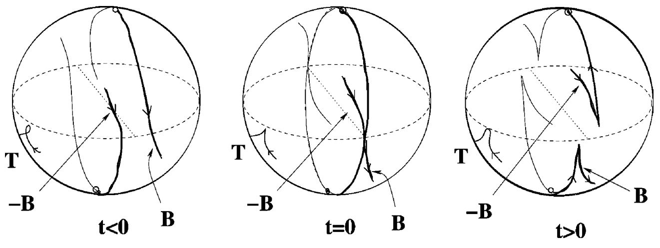

Fig. 3. Perestroika of the caustics (by the Gauß map) of curves in $\mathbb{R}^3$ in a generic one-parameter family during an inflection perestroika. This caustic consists of $\pm\mathbf{B}$ the binormal indicatrix.

The Darboux indicatrix is a caustic: it is formed by the singular points of all equidistant fronts to the tangent indicatrix (or to the binormal indicatrix). A point of an immersed curve in the Euclidean space $\mathbb{R}^3$ is called an *inflection* of the curve if the first and second derivatives of the curve at that point are linearly dependent (at an inflection the curvature vanishes).

A generic curve has no inflection. However, for generic one-parameter families of curves in $\mathbb{R}^3$ (we look upon the parameter as the time), at isolated parameter values, two events (called *perestroikas*) may occur at which the number of flattenings changes: a biflattening or an inflection of the curve. At the moment of a biflattening or inflection, two close flattenings of the curve are born or killed; in each of these perestroikas, twistings play an important role.

It is interesting to note that under the inflection bifurcation described in [9,10], the Legendre curve associated with the Frenet frame experience global topological bifurcations. This kind of non-local bifurcations have never been studied before in singularity theory.

As a consequence, the tangent indicatrix, the binormal indicatrix, their caustic (i.e. the Darboux indicatrix) and the normal indicatrix experience global topological bifurcations (see Figs. 2–5).

Remark. However, we must say that Shcherbak [8] has studied the perestroikas of the front and of the tangent developable of a curve in the space $\mathbb{R}P^3$: the front is a surface—in the dual space—which is formed by all the planes tangent to the curve; the tangent developable is the surface formed by the points of all the tangent lines to the curve. Shcherbak has

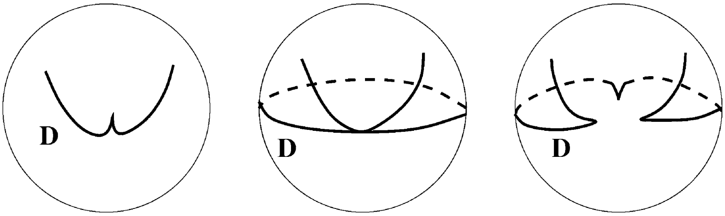

Fig. 4. Darboux indicatrix during an inflection perestroika.

proved that at an inflection perestroika the cuspidal edge of the front and the cuspidal edge of the tangent developable both experiment a global perestroika. However, he has not studied if there is (or not) a perestroika of the corresponding Legendrian manifolds. In [7], Mond has studied—independently and with other methods—the bifurcation of the tangent developable of a curve in $\mathbb{R}P^3$ at an inflection perestroika.

Usually, in the study if Legendrian singularities one considers one-parameter families of smooth Legendrian submanifolds which depends smoothly on the parameter; in particular, all the Legendrian submanifolds of the family are diffeomorphic (one to each other). One considers a Legendrian projection and studies the bifurcations of the corresponding one-parameter families fronts. However, in the study of generic one-parameter families of curves in the Euclidean 3-space one can obtain in a natural way a one-parameter family of Legendrian curves having a perestroika. Note that the Legendrian curve in $ST^*\mathbb{S}^2$ associated to a curve in the Euclidean 3-space is singular (see Fig. 6). This explains why the 'non-standard' perestroikas of caustics (Fig. 4) and fronts (Figs. 2 and 3) in one-parameter families appear in this situation. The normal indicatrix of a closed curve in the Euclidean 3-space is an (exact) Lagrangian curve of the sphere with the standard symplectic structure. In an inflection perestroika, these Lagrangian curves (i.e. the normal indicatrices) also experiment a global perestroika (Fig. 5).

Consider the Frenet frame of a curve in the Euclidean 3-space. The front given by the first vector of the frame (i.e. the tangent indicatrix) may have spherical inflections generically, but generically it has no cusp. The front given by the third vector (i.e. the binormal indicatrix) may have cusps generically, but generically it has no spherical inflection.

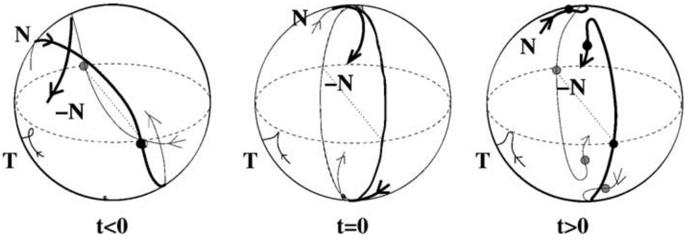

Fig. 5. Perestroika of the family of normal indicatrices of a generic one-parameter family of curves in $\mathbb{R}^3$ having an inflection perestroika.

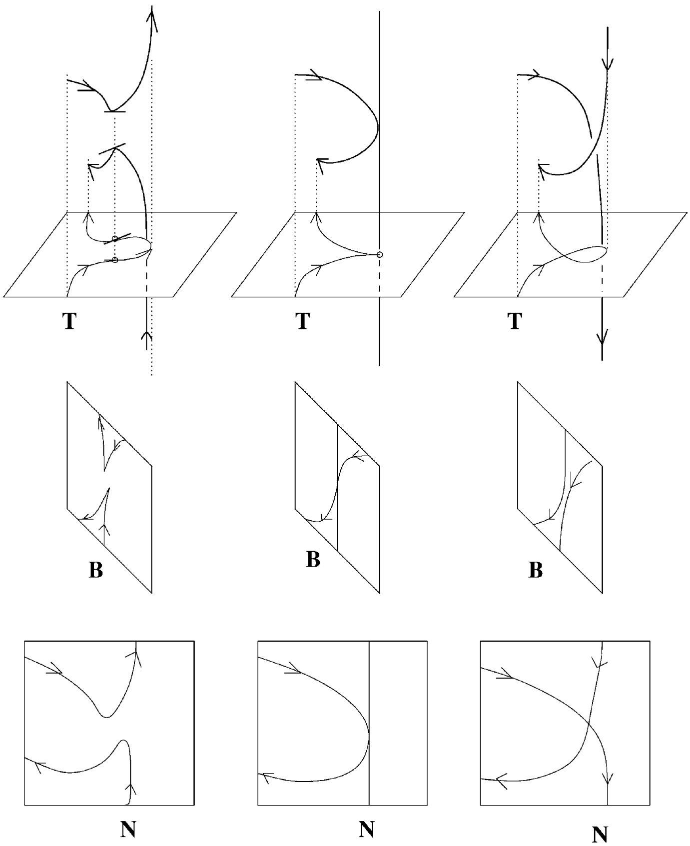

Fig. 6. The three standard projections of the Legendrian curve associated to the tangent indicatrix of a space curve. The letters **T**, **B** and **N** denote the tangent, normal and binormal indicatrix, respectively.

One can impose the condition that the Legendrian curve in the space of orthonormal frames is smooth and smoothly depends on parameters. Then the non-local bifurcation described above is not possible in this case. Instead, both fronts (given by the first and the third vectors of the frame) may have cusps generically and may have spherical inflections generically.

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