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The effect of trapped charge on series capacitors

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If capacitors are initially charged before placing them in series, charge becomes trapped on the electrically isolated internal plates. The effect of this "trapped charge" on the final charge and voltage distributions in series capacitor networks provides instructors with a new class of engaging capacitor problems not currently addressed in introductory physics textbooks. We present formulae for the final charges on two series capacitors connected to a battery in terms of initial charge values. Various special cases are also considered. Results are verified experimentally using dc voltage and *RC* time constant measurements. Practical considerations for experimental design are discussed.
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I. INTRODUCTION

"How do the charges on two unequal capacitors ($C_1 > C_2$) connected in parallel to a battery compare to the charges on the capacitors when they are instead connected in series to the battery?" This question[1] is routinely posed in various forms to introductory physics classes to illustrate the conclusion that charges on dissimilar capacitors connected in series must be equal. The following excerpt from Serway[2] (modified slightly to utilize our Fig. 1) is typical of textbook explanations:

> *The top plate of C_1 and the bottom plate of C_2 are connected to the terminals of a battery. The two inner plates are connected to each other and to nothing else; hence, they form an isolated system that is initially uncharged and must continue to have zero net charge. To analyze this combination, let's first consider the uncharged capacitors and then follow what happens immediately after a battery is connected to the circuit. When the battery is connected, electrons are transferred out of the top plate of C_1 and into the bottom plate of C_2. As this negative charge accumulates on the bottom plate of C_2, an equivalent amount of negative charge is forced off the top plate of C_2, and this top plate therefore has an excess positive charge. The negative charge leaving the top plate of C_2 causes negative charge to accumulate on the bottom plate of C_1. As a result, both bottom plates end up with a charge of $-Q$ and both top plates end up with a charge of $+Q$. Therefore, the charges on capacitors connected in series are the same: $Q_1 = Q_2 = Q$.*

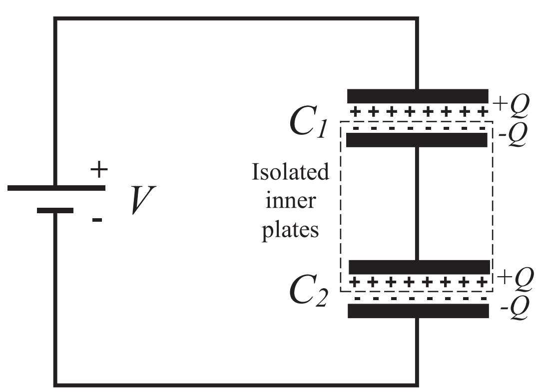

Fig. 1. In the familiar textbook example, C_1 and C_2 are initially uncharged when they are placed in series with a battery. Electrons on the bottom plate of C_1 must necessarily be pulled from the top plate of C_2 because the inner plates of the series combination are electrically isolated.

When this question was posed to our class recently, a student asked an interesting question that we never considered: "Are the capacitors connected in parallel to the battery to be discharged before they are subsequently connected in series with the battery?" All of the introductory physics and electronics textbooks we surveyed derive conclusions about charges for series capacitors based on the assumption that the capacitors connected in series are initially uncharged.[1–16] But, if we allow capacitors to possess nonzero initial charge before we place them into a series combination, we discover an intriguing and delightfully counterintuitive effect of the charge that becomes trapped on the electrically isolated internal plates.

II. DERIVATION OF FINAL CAPACITOR CHARGES

To find the relative charges on capacitors in series with a battery, we first consider the arbitrary situation illustrated in Fig. 2(a), where, before hooking the capacitors in series, C_1 and C_2 are initially charged to Q_1^{init} and Q_2^{init}, respectively. Even though the two inner plates are connected in Fig. 2(a), no charge will flow between the capacitors until the circuit is completed by the closing of the switch as in Fig. 2(b). Our subsequent discussion is based on the initial polarities shown in Fig. 2, where the negative plate of C_1 is connected to the positive plate of C_2 before the switch is closed. We demonstrate how our equations may be applied to situations with different initial polarities in Sec. III D.

Note that the arrangements of charge depicted in Fig. 2, where each capacitor possesses equal but opposite charges distributed uniformly along its internal faces, are idealized. Other authors have investigated the effects of fringing, dielectrics, and energy considerations on the actual charge distribution of non-idealized series capacitors.[17–20] Such considerations, however, result in small deviations from the idealized condition and are tangential to the focus of this article.

Figure 2(b) illustrates the final distribution of charges Q_1 and Q_2 after the switch is closed. Because the inner plates of C_1 and C_2 are electrically isolated, the net charge trapped on these plates before and after the switch is closed must remain constant

$$Q_{\text{trap}} = Q_2^{\text{init}} - Q_1^{\text{init}} = Q_2 - Q_1. \tag{1}$$

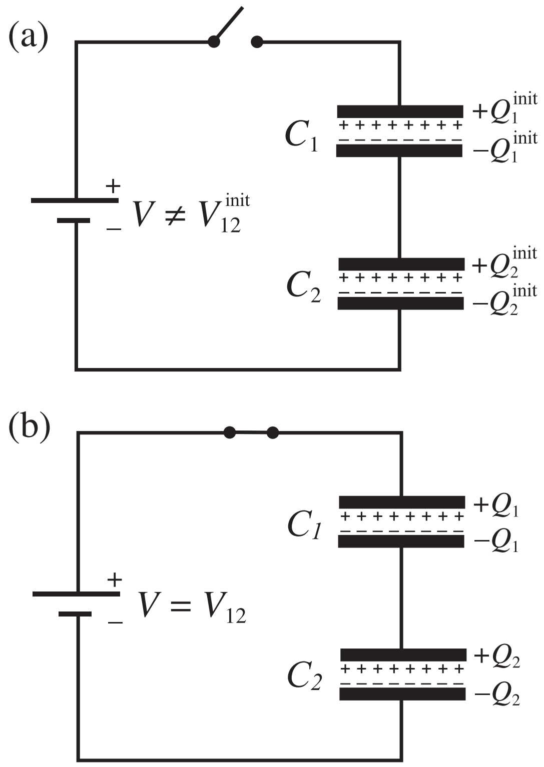

Fig. 2. Capacitors C_1 and C_2 possess arbitrary initial charges (a). Because the switch is open, the battery voltage is not in general equal to the sum of the initial capacitor voltages: $V \neq V_1^{init} + V_2^{init}$. When the switch closes (b), the capacitors are placed in series and charge flows in the circuit until the combined capacitor voltages become equal to the battery voltage: $V = V_1 + V_2$.

We obtain a second independent equation by applying Kirchhoff's loop rule and the definition of capacitance ($Q = CV$) after the switch is closed, which yields

$$V - \frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0. \tag{2}$$

Solving Eqs. (1) and (2) simultaneously, we obtain the final charges on each capacitor in terms of the initial charges and battery voltage (V) to be

$$Q_1 = \frac{C_1C_2}{C_1 + C_2}V - \frac{C_1}{C_1 + C_2}\left(Q_2^{init} - Q_1^{init}\right) \tag{3}$$

and

$$Q_2 = \frac{C_1C_2}{C_1 + C_2}V + \frac{C_2}{C_1 + C_2}\left(Q_2^{init} - Q_1^{init}\right). \tag{4}$$

Interestingly, we see that the charges on the series capacitors are only equal if there is no trapped charge on the electrically isolated internal plates ($Q_2^{init} - Q_1^{init} = 0$). We can thus conclude that the ubiquitous conclusion found in introductory physics textbooks that, "charges on capacitors placed series must be equal," is only valid when the capacitors under consideration are initially uncharged or they have equal initial charges. In the trivial case where the battery voltage is equal to the initial combined voltage across the two capacitors ($V = V_1^{init} + V_2^{init}$), no charge will flow in the circuit when the switch is closed and Eqs. (3) and (4) reduce to $Q_1 = Q_1^{init}$ and $Q_2 = Q_2^{init}$.

In the standard situation where the two capacitors are initially uncharged and placed in series with a battery, they would each acquire a final charge $Q = [C_1C_2/(C_1 + C_2)]V$. Thus, Eqs. (3) and (4) can be alternately written as

$$Q_1 = Q - \frac{C_1}{C_1 + C_2}Q_{trap} \tag{5}$$

and

$$Q_2 = Q + \frac{C_2}{C_1 + C_2}Q_{trap}. \tag{6}$$

Next, we consider various special cases to gain conceptual insight into the effect of trapped charge on the charge distribution of series capacitors. Since voltages are easier to measure than stored charges, it will be convenient to rewrite Eqs. (3) and (4) in terms of voltages as

$$V_1 = \frac{C_2}{C_1 + C_2}V + \frac{C_1V_1^{init} - C_2V_2^{init}}{C_1 + C_2} \tag{7}$$

and

$$V_2 = \frac{C_1}{C_1 + C_2}V + \frac{C_2V_2^{init} - C_1V_1^{init}}{C_1 + C_2}. \tag{8}$$

Interestingly, we see that the capacitor voltages depend on the initial voltages of each capacitor that existed before they were placed in the series circuit (in addition to C_1, C_2, and V). In other words, the voltages on each capacitor in a series combination with trapped charge retain a "memory" of the initial capacitor voltages.

III. SPECIAL CASES

A. Capacitors with equal initial charges

The familiar result for initially uncharged capacitors is confirmed by substituting $V_1^{init} = V_2^{init} = 0$ into Eqs. (7) and (8) to find

$$V_1 = \frac{C_2}{C_1+C_2}V \tag{9}$$

and

$$V_2 = \frac{C_1}{C_1+C_2}V. \tag{10}$$

Here, we observe that for $C_1 \neq C_2$, the larger capacitor ends up having the smaller voltage as is commonly taught in introductory classes. Testing this result experimentally is not as straightforward as one might imagine. In making measurements to confirm our theoretical predictions for this study, we sought to use typical capacitors that would be readily available, and arbitrarily chose $C_1 = 1.980\,\mu$F and $C_2 = 1.036\,\mu$F, as measured with a Wavetek capacitance meter. All voltmeters have some input impedance (typically 1–10 MΩ) and if one naively attempts to measure the capacitor voltages there results the situation depicted in Fig. 3(a). The input impedance of the voltmeter in Fig. 3(a) acts as a shunt resistor that allows negative charge to flow from the junction between the negative plate of C_2 and the battery to the junction between the two capacitors, simultaneously

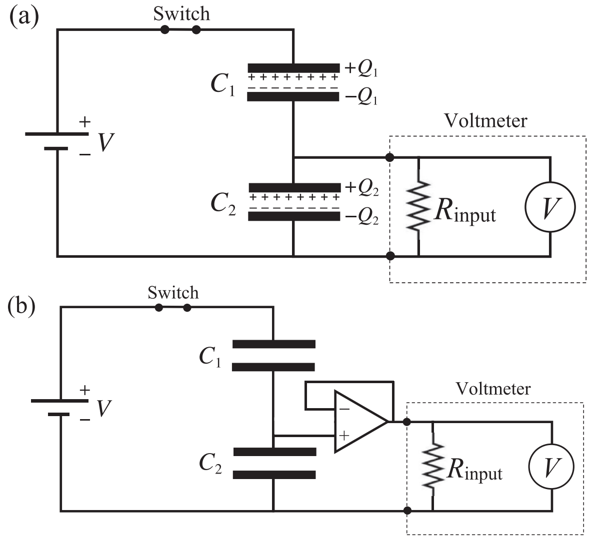

Fig. 3. In (a), the input impedance of a typical voltmeter acts as a shunt resistance causing the measurement of V_2 to decay during measurement. In (b), an operational amplifier buffers the voltmeter to allow for stable voltage measurements. Note that care must be taken to make sure the capacitor voltages being measured with the configuration in (b) do not exceed the ± supply voltages of the op-amp.

draining C_2 and charging C_1. We find the RC time constant in Fig. 3(a) for the discharging of C_2 (and corresponding charging of C_1) to be 3.0 s using our lab voltmeter and the capacitor values we selected. Consequently, if one uses such capacitors and attempts to make voltage measurements using the circuit depicted in Fig. 3(a) with a typical voltmeter, the voltage V_2 will be seen to decay with time. Similar results would be observed using a Logger-Pro voltage probe, whose input impedance is 10 MΩ.

An experimenter attempting to observe static voltages on series capacitors must then choose to either drastically increase the capacitor values or the input impedance of the voltmeter. If we increase capacitance values by using expensive 1-F capacitors, we increase the RC time constant of the circuit depicted in Fig. 3(a) to a very stable $\tau = 35$ days. However, we feel that using these extraordinary 1-F capacitors might give students the impression that results are only valid in extreme conditions. We prefer instead to increase the voltmeter input impedance by buffering our voltmeter with a JFET operational amplifier (op-amp) hooked up in a "voltage follower" configuration as shown in Fig. 3(b).[21] Using a Texas Instruments LF-356 op-amp, we effectively increase the input impedance of our voltmeter to well over 10^{11} Ω, resulting in observed voltages that are completely stable over the time required to make voltage measurements.

Figure 4 shows experimental observations of V_1 using the $Q_1^{\text{init}} = Q_2^{\text{init}} = 0$ initial condition compared to theoretical predictions from Eq. (9) for various battery voltages. All observed voltages were extremely reproducible and within 1% of theoretical predictions.

Equations (9) and (10) are also valid when the initial charges on the capacitors are equal and non-zero: $Q_1^{\text{init}} = Q_2^{\text{init}} \neq 0$. Using the definition of capacitance, the equal-initial-charges condition requires that the initial capacitor voltages obey the relation

$$V_2^{\text{init}} = \frac{C_1}{C_2} V_1^{\text{init}}. \tag{11}$$

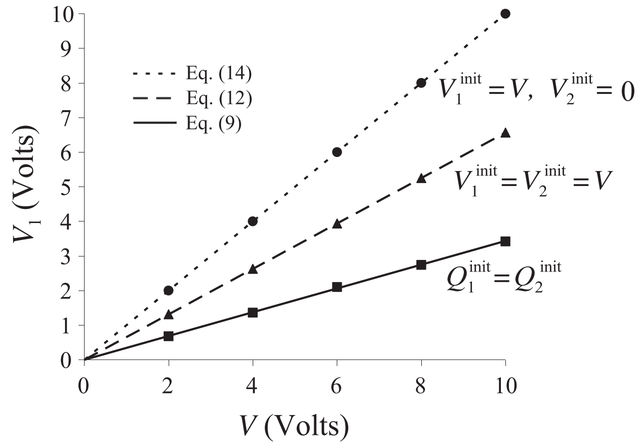

Fig. 4. Experimental results for three special cases with initial conditions described in the text are plotted against theoretical curves. Each data point was obtained through the process portrayed in Fig. 2, whereby both capacitors were charged to their respective initial voltages before the switch was closed. Capacitance values for all trials were $C_1 = 1.980\,\mu$F, and $C_2 = 1.036\,\mu$F.

For each of the five battery voltages in used in Fig. 4, we ran additional sets of trials to create identical nonzero charges on the capacitors, first charging C_1 to some arbitrary voltage, and then charging C_2 to the corresponding initial voltage given by Eq. (11). In each case, the final capacitor voltages after the switch was closed were equal (within 1%) to those observed for our previous set of measurements for the initially uncharged capacitors. We can understand this by noting that for each trial, the net charge that is trapped $(Q_2^{\text{init}} - Q_1^{\text{init}})$ on the isolated internal plates of C_1 and C_2 must be zero since the initial charges are equal. When the switch is closed, the power supply simply removes or adds the appropriate charges to the outer plates to bring the series combination to the power supply voltage, and the zero net charge on the inner plates rearranges itself accordingly. The data points from our $Q_1^{\text{init}} = Q_2^{\text{init}} \neq 0$ measurements are not plotted in Fig. 4 since they fall on top of the previous data points from our trials with $Q_1^{\text{init}} = Q_2^{\text{init}} = 0$.

B. Capacitors initially charged to supply voltage

We now consider the case posed by the student's question that inspired this manuscript. When each capacitor is initially charged to the battery voltage before placing the capacitors in series ($V_1^{\text{init}} = V_2^{\text{init}} = V$), Eqs. (7) and (8) reduce to

$$V_1 = \frac{C_1}{C_1 + C_2} V \tag{12}$$

and

$$V_2 = \frac{C_2}{C_1 + C_2} V. \tag{13}$$

Interestingly, in this special case where each capacitor is initially charged to the battery voltage, we see that the larger capacitor experiences the larger voltage in the same way that the larger resistor in a series combination of resistors experiences the larger voltage in the familiar voltage divider equation for series resistors.[22] Figure 4 shows experimental observations of V_1 compared to theoretical predictions from Eq. (12) for various battery voltages.

C. C_1 initially charged to battery voltage and C_2 initially uncharged

Another interesting case is when C_1 is initially charged to V and then connected in series with an initially uncharged C_2 ($V_1^{\text{init}} = V$ and $V_2^{\text{init}} = 0$). Substituting these values into Eqs. (7) and (8), we obtain

$$V_1 = V \tag{14}$$

and

$$V_2 = 0. \tag{15}$$

In this case, no charge is transferred to C_2 when the switch in Fig. 2 is closed. Figure 4 shows experimental observations of V_1 compared to theoretical predictions from Eq. (14) for various battery voltages.

D. Zero battery voltage

While none of the textbooks we surveyed[1–16] considered the problem of placing initially charged capacitors in series with a battery, nearly all of these texts posed various exercises challenging students to find the final voltage of two capacitors that are initially charged and then connected to one another without a battery. Figure 5 illustrates this example with the common polarities of the capacitors joined by the closing of two switches as the textbook problem is typically posed. The second switch in Fig. 5 is superfluous since no charge will flow unless both switches are closed.

We may apply Eqs. (7) and (8) to this problem by making two substitutions based on a comparison between Figs. 2 and 5. First we note that the battery in Fig. 2 has been replaced by a wire in Fig. 5. We accommodate this feature in Fig. 5 by substituting $V = 0$ in Eqs. (7) and (8) since a wire can be considered a battery of zero potential. We also note that instead of connecting the negative plate of C_1 to the positive plate of C_2, as in Fig. 2, the switch in Fig. 5 connects the negative plate of C_1 to the negative plate of C_2. We accommodate this difference by adding a negative subscript to the initial voltage on C_2 to indicate that the polarity has been reversed ($V_{2-}^{\text{init}} = -V_2^{\text{init}}$) from that used in Fig. 2 and subsequent derivations. So, for example, if the initial voltage on C_2 in Fig. 5 was $V_{2-}^{\text{init}} = +20\,\text{V}$, we would substitute $V_2^{\text{init}} = -20\,\text{V}$ in Eqs. (7) and (8). Making these substitutions, the solution to the textbook problem is then

$$V_1 = \frac{C_1 V_1^{\text{init}} + C_2 V_{2-}^{\text{init}}}{C_1 + C_2}. \tag{16}$$

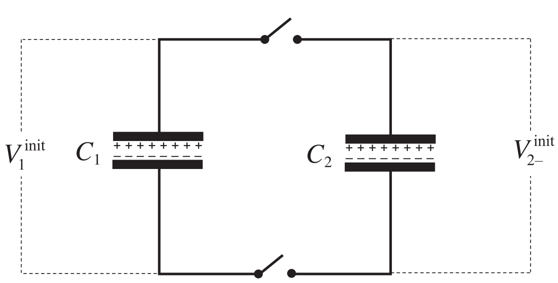

Fig. 5. The example of connecting two initially charged capacitors that is found in most introductory university physics textbooks. The initial voltage on C_2 is labeled with a minus subscript to indicate that its polarity is reversed from that in Fig. 2, where $V_{2-}^{\text{init}} = -V_2^{\text{init}}$.

and

$$V_2 = -V_1. \tag{17}$$

We can interpret the minus sign in Eq. (17) as a result of the voltage measurement perspective in Fig. 2, where we place the positive lead of an imaginary voltmeter at the junction between the capacitors (at the negative plate of C_1) when measuring V_2.

The most common solution to this problem requires noting that the final capacitor voltages must be equal, since the capacitors are effectively in parallel ($V_1 = V_{2-}$), and that the net charges on joined capacitor faces must be conserved. This approach yields an identical result to Eqs. (16) and (17).

IV. THE OVERALL CAPACITANCE OF SERIES CAPACITORS WITH TRAPPED CHARGE

Does the presence of trapped charge in series capacitors influence the overall capacitance of the series combination? In order to answer this question, we must first review how the capacitance of series capacitors with no initial charge is typically determined. In so doing, we will see why the usual textbook definition of capacitance cannot be applied directly to this situation.

Introductory physics texts introduce the concept of capacitance by considering the electric potential difference that results from placing charges $+Q$ and $-Q$ on opposing conductors. This passage from Serway and Jewett[23] is emblematic of this approach: "The capacitance C of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between the conductors: $C = Q/\Delta V$." Applying this approach to a simple series combination of initially uncharged capacitors C_1 and C_2, all introductory physics textbooks report the capacitance of the series combination to be

$$C_{\text{series}} = \frac{C_1 C_2}{C_1 + C_2}. \tag{18}$$

However, if we attempt to apply the standard textbook definition of capacitance to a series combination of initially charged capacitors we are immediately faced with a difficulty. Unlike initially uncharged conductors, the series combination of capacitors with initial charge in Fig. 2(a) already possesses unequal charges $+Q_1^{\text{init}}$ and $-Q_2^{\text{init}}$ on the outer two capacitor faces. We might attempt to simply redefine Q in the traditional definition of capacitance as the magnitude of *additional* charge that could be placed on each of the outer two capacitor faces, however, a simple physical mechanism for accomplishing this task is not immediately obvious.

Keeping in mind that the capacitance of an ordinary parallel-plate capacitor is independent of charge and voltage, we can write

$$dQ = C\,dV, \tag{19}$$

which means the definition of capacitance can be written in differential form as

$$C = \frac{dQ}{dV}. \tag{20}$$

It follows from Eq. (20) that we can think of capacitance as the ratio of the change in charge on the capacitor to some change in voltage that has been imposed on the capacitor. For example, consider a parallel-plate capacitor that has been charged to a voltage of 10 V. If we observe that the capacitor draws an additional charge of 6 μC when its voltage is increased to 12 V, we can conclude that its capacitance is $C = \Delta Q/\Delta V = 6\,\mu\mathrm{C}/2\,\mathrm{V} = 3\,\mu\mathrm{F}$.

To apply Eq. (20) to a series combination of capacitors with trapped charge, we begin by taking the differentials of Eqs. (3) and (4) to get the change in charge for each capacitor that would result from a change in battery voltage dV

$$dQ_1 = dQ_2 = \frac{C_1 C_2}{C_1 + C_2}\,\mathrm{dV}. \tag{21}$$

Like a traditional series capacitor circuit with no trapped charge, the only part of this change in charge that is recoverable is that which resides on the outer plates of the series combination. Thus, as in the traditional case of initially uncharged series capacitors that experience a change in battery voltage, the change in recoverable charge dQ for the series combination is not the sum $dQ_1 + dQ_2$, but rather $dQ = dQ_1 = dQ_2$. We can then integrate Eq. (21) to find the charge $Q = \Delta Q_1 = \Delta Q_2$ that is transferred to the series combination when the battery voltage is changed

$$Q = \frac{C_1 C_2}{C_1 + C_2}\,\Delta V. \tag{22}$$

With Eq. (22), we see now that the overall capacitance of series capacitors with trapped charge is identical to that of uncharged series capacitors given in Eq. (18). To test Eq. (22) experimentally, we first imagine replacing the battery in Fig. 2(a) with a variable dc power supply. This variable voltage acts as a battery whose voltage can be adjusted to any arbitrary voltage. Before we close the switch in Fig. 2(a) we adjust the variable voltage to initially be exactly equal to the sum of initial capacitor voltages, that is, $V^{\mathrm{init}} = V_{12}^{\mathrm{init}} = V_1^{\mathrm{init}} + V_2^{\mathrm{init}}$. When we close the switch as in Fig. 2(b), no current flows in the circuit and the initial capacitor voltages remain unchanged. As we then change the battery voltage to some new value, the individual capacitor charges at any time can be found by adding the charge Q in Eq. (22) to the initial charges:

$$Q_1 = Q_1^{\mathrm{init}} + Q \tag{23}$$

and

$$Q_2 = Q_2^{\mathrm{init}} + Q. \tag{24}$$

We can likewise rewrite these equations in terms of voltages as

$$V_1 = V_1^{\mathrm{init}} + \frac{C_2}{C_1 + C_2}\,\Delta V \tag{25}$$

and

$$V_2 = V_2^{\mathrm{init}} + \frac{C_1}{C_1 + C_2}\,\Delta V. \tag{26}$$

In a typical laboratory situation, it would be very difficult to match the initial power supply voltage (V^{init}) exactly to the sum of initial capacitor voltages (V_{12}^{init}) as in the experimental design described in the previous paragraph. It is more convenient to start with our initial power supply voltage set to some arbitrary voltage that is not equal to V_{12}^{init}. If we then start our experiment to validate Eqs. (25) and (26) with $V^{\mathrm{init}} \neq V_{12}^{\mathrm{init}}$ as suggested, we see that when we first attach the capacitors with trapped charge to our power supply, the capacitor voltages will quickly equilibrate to new values, V_1' and V_2', given by Eqs. (7) and (8), just as if we had closed a switch as in Fig. 2. The initial voltages on the capacitors in Eqs. (25) and (26) for the typical laboratory situation then become $V_1^{\mathrm{init}} = V_1'$ and $V_2^{\mathrm{init}} = V_2'$. We may then observe how the capacitor voltages change as we subsequently adjust the variable power supply voltage. We applied this approach using initial conditions from Sec. III B with $V_1^{\mathrm{init}} = V_2^{\mathrm{init}} = V = -10$ V to obtain the data presented in Fig. 6. Once the circuit was connected, the capacitor voltages immediately equilibrated to new values, given by Eqs. (7) and (8). The subsequent capacitor voltages plotted in Fig. 6 were measured continuously as the power supply voltage was slowly ramped from -10 V to $+10$ V. The theoretical relationships plotted in Fig. 6 are found from Eqs. (25) and (26) using the ordinary definition of change in battery voltage, $\Delta V = V^{\mathrm{final}} - V^{\mathrm{init}}$.

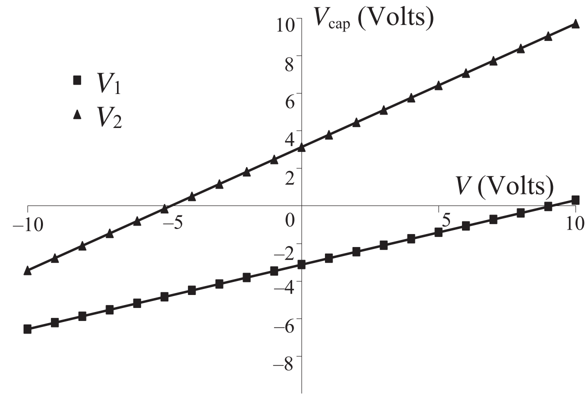

Fig. 6. After charging each capacitor to -10 V, the capacitors were hooked in series to a variable power supply whose voltage V was then slowly ramped from -10 V to $+10$ V. Capacitor voltages were measured continuously as V was ramped. The theoretical curves were obtained from Eqs. (25) and (26) with initial capacitor voltages obtained from Eqs. (7) and (8). Note that while the starting voltages in this figure agree with Eqs. (12) and (13) for the special condition $V_1^{\mathrm{init}} = V_2^{\mathrm{init}} = V$, this relationship does not continue to hold as the battery voltage is ramped; this is because the capacitors are not recharged to meet the special initial conditions before each voltage measurement, as in the procedure used to obtain Fig. 4.

Note that, in our above discussions, we have assumed that the changes to power supply voltage are slow compared to the time constant of the battery/capacitor circuit. Otherwise, there would have been a time lag, as we discuss in Sec. V.

V. *RC* TIME CONSTANT WITH TRAPPED CHARGE

We see from Eq. (22) that series capacitors with trapped charge yield the same recoverable charge as initially uncharged capacitors when subjected to a change in battery voltage. But, does the presence of trapped charge influence the time it takes to discharge an *RC* circuit with series capacitors? Consider the series *RC* circuit in Fig. 7. When the switch in Fig. 7 is closed, charge will flow through R until the combined voltage V_{12} across the capacitors is equal to the battery voltage. This is analogous to the situation

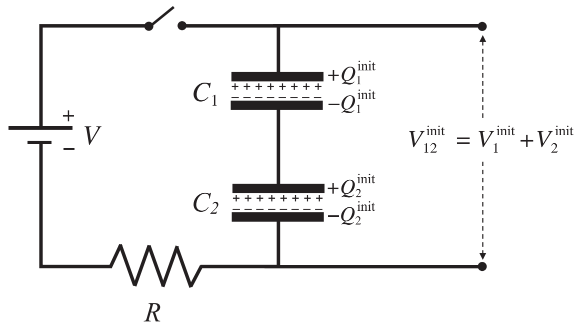

Fig. 7. Series capacitors with initial charges are connected in an *RC* circuit with a battery. When the switch is closed, the combined capacitor voltage V_{12} goes from $V_{12}^{\rm init}$ to V with the same time constant observed for capacitors without initial charge.

portrayed in Fig. 2, only now the charging process is slowed by the presence of a resistor.

As is usually done for *RC* circuits,[24] we can write Kirchhoff's loop rule for the circuit in Fig. 7 when the switch is closed as

$$V - R\frac{dQ}{dt} - \frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0. \tag{27}$$

Substituting Eqs. (23) and (24) into Eq. (27), we eliminate Q_1 and Q_2 and obtain the following differential equation for the charge Q transferred to the series capacitor combination:

$$\left(V - V_{12}^{\rm init}\right) - R\frac{dQ}{dt} - \left(\frac{1}{C_1} + \frac{1}{C_2}\right)Q = 0. \tag{28}$$

Equation (28) is identical to the traditional equation obtained for initially uncharged capacitors except for the constant term $V_{12}^{\rm init}$. As with an ordinary *RC* circuit, constant terms in Eq. (28) will not impact the *RC* time constant obtained from the solution. We conclude that the time constant of an *RC* circuit with series capacitors containing trapped charge is identical to that of a circuit with series capacitors without trapped charge.

We tested the above result experimentally by placing our series capacitor combination in the circuit shown in Fig. 7, with $R = 1.571\,\text{k}\Omega$. We replaced the battery and switch with a function generator set to create a 5 V peak-to-peak square wave with a dc offset of +5 V so that it provided a pulsed dc waveform of sufficient period to measure the *RC* time constant of the circuit with an oscilloscope. The output impedance of our function generator ($50\,\Omega$) was sufficiently smaller than R that we could safely disregard its influence in our time constant measurement. A buffer amplifier was not required for our oscilloscope in parallel with V_{12} because the 1-MΩ input impedance of the oscilloscope would not produce any noticeable discharging of the capacitors on the short time scale of our measurement.

For our first *RC* time constant measurement, we used the initial conditions described in Sec. III C with $V_1^{\rm init} = +10\,\text{V}$ and $V_2^{\rm init} = 0$. We measured a time constant τ of 1.1 ms for this series capacitor combination with trapped charge, which corresponds to a total capacitance of $C = \tau/R = 0.70\,\mu\text{F}$. Given the inherent imprecision of making voltage measurements from an oscilloscope display, this value is in acceptable agreement with the ordinary series capacitance of $0.6801\,\mu\text{F}$ from Eq. (18). The capacitors were then discharged and we measured an identical time constant of 1.1 ms for the initially uncharged series capacitors, with no perceptible difference in the oscilloscope trace. We then measured the time constant for the initial conditions described in Sec. III B with $V_1^{\rm init} = V_2^{\rm init} = +10\,\text{V}$. As with our previous measurements, the time constant for this initial condition was also measured to be 1.1 ms.

VI. CONCLUSIONS

Introductory physics textbooks demonstrate that charges on series capacitors must be equal based on an assumption that the capacitors are initially uncharged before placing them in a circuit with a battery. Noting that the net charge on the electrically isolated inner plates of a series capacitor combination is constant, we applied Kirchhoff's loop rule to derive equations for the final voltages of two series capacitors that each possessed initial charge. We then measured voltages on one of the capacitors for three special initial charge conditions and demonstrated good agreement with our theoretical predications.

We also showed that when the overall voltage of a series combination of capacitors with initial charge is changed, the amount of charge that flows to (or from) the capacitors does not depend on their initial charges. Therefore, the equivalent capacitance (as defined by $C_{\rm eq} = \Delta Q/\Delta V$) for a series combination of capacitors with initial charge is the same as it would be with no initial charge. This result was then tested experimentally and shown to agree with this prediction. Lastly, we showed that the *RC* time constant of a circuit with two series capacitors does not depend on the initial charges on the capacitors.

We have presented an analysis for the simplest case of only two series capacitors. Instructors may apply our techniques to formulate more advanced problems that utilize more complex capacitor networks.

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[1]This question was adapted from the question 13-14 sequence on page 801, R. Serway and J. Jewett, *Physics for Scientists and Engineers*. 9th ed. (Brooks/Cole, Pacific Grove, Ca., 2013), Chap. 26.

[2]See Ref. 1, p. 784.

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[24] See, for example, Ref. 4, pp. 864–868.

Student Wheatstone Bridge

This small student Wheatstone bridge was a bread and butter item for Leeds & Northrup of Philadelphia, the maker of precision instruments for electrical measurements. This example, in the Greenslade Collection, was sold to Wellesley College in 1917. Ten years later, it was given push-buttons for bringing the battery and then the galvanometer into the circuit, and was listed as an "Enclosed-Dial Wheatstone Bridge" and sold for $55.00. In 1914 L&N had patented the enclosed dial resistance unit, which made the instrument more useful for hostile environments and eliminated the need to clean the switch contacts that were exposed in earlier models. A surprising number of these bridges have survived into the 21st century! (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College.)