

VIŠESTRUKI INTEGRALI - ZADACI (III DEO)

Izračunavanje površine u ravni primenom dvostrukog integrala

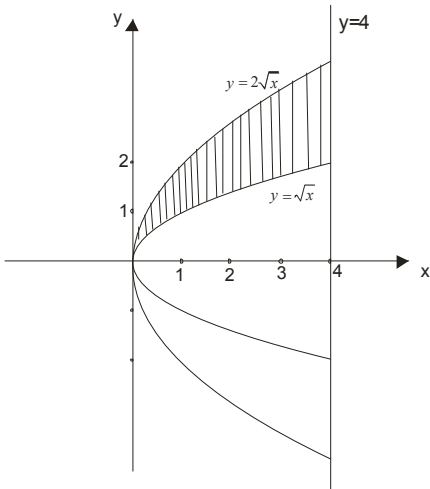
Površina oblasti D u ravni xOy može se naći po formuli:
$$P = \iint_D dx dy$$

Primer 1.

Izračunaj površinu ograničenu sledećim linijama: $y = \sqrt{x}$, $y = 2\sqrt{x}$ i $y = 4$.

Rešenje:

Najpre ćemo, kao i uvek, nacrtati sliku i odrediti granice po kojim radimo...



Oblast integracije je osenčena na slici $D: \begin{cases} 0 \leq x \leq 4 \\ \sqrt{x} \leq y \leq 2\sqrt{x} \end{cases}$

Upotrebom gore navedene formule, računamo površinu osenčenog dela:

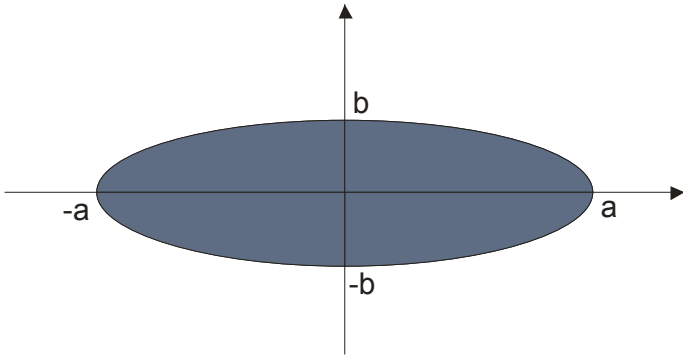
$$\begin{aligned} P &= \iint_D dx dy = \int_0^4 \left(\int_{\sqrt{x}}^{2\sqrt{x}} dy \right) dx = \int_0^4 \left(y \Big|_{\sqrt{x}}^{2\sqrt{x}} \right) dx = \int_0^4 (2\sqrt{x} - \sqrt{x}) dx = \int_0^4 \sqrt{x} dx = \\ &= \int_0^4 x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^4 = \frac{2}{3} \left(4^{\frac{3}{2}} - 0^{\frac{3}{2}} \right) = \frac{2}{3} \cdot 8 = \boxed{\frac{16}{3}} \end{aligned}$$

Primer 2.

Izračunaj površinu ograničenu sa $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Rešenje:

Naravno, ovde je u pitanju elipsa. Mi treba da izračunamo površinu unutar nje...



Ovde je zgodno uzeti takozvane **eliptičke koordinate**:

$$x = a r \cos \varphi$$

$$y = b r \sin \varphi \quad \text{onda je:} \quad \iint_D z(x, y) dx dy = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^r z(ar \cos \varphi, br \sin \varphi) abr dr$$

$$|J| = abr$$

Da vidimo zašto su ove smene dobre:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{(ar \cos \varphi)^2}{a^2} + \frac{(br \sin \varphi)^2}{b^2} = 1$$

$$\frac{\cancel{a^2} r^2 \cos^2 \varphi}{\cancel{a^2}} + \frac{\cancel{b^2} r^2 \sin^2 \varphi}{\cancel{b^2}} = 1$$

$$r^2 \cos^2 \varphi + r^2 \sin^2 \varphi = 1$$

$$r^2 (\cos^2 \varphi + \sin^2 \varphi) = 1$$

$$r^2 = 1 \rightarrow r = 1$$

Dobijamo da je $0 \leq r \leq 1$, pošto ugao uzima ceo krug, to je $0 \leq \varphi \leq 2\pi$.

$$\text{Oblast } D : \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

Sad rešavamo dvojni integral :

$$P = \iint_D dx dy = \int_0^{2\pi} d\varphi \int_0^1 ab r dr = ab \int_0^{2\pi} \left(\frac{r^2}{2} \right) \Big|_0^1 d\varphi = \frac{1}{2} ab \int_0^{2\pi} d\varphi = \frac{1}{2} ab \cdot 2\pi = \boxed{ab\pi}$$

Površina elipse se dakle računa po formuli $P = ab\pi$

Primer 3.

Izračunaj površinu ograničenu sledećim linijama: $x^2 + y^2 = 2x$, $y = x$ i $y = 0$

Rešenje:

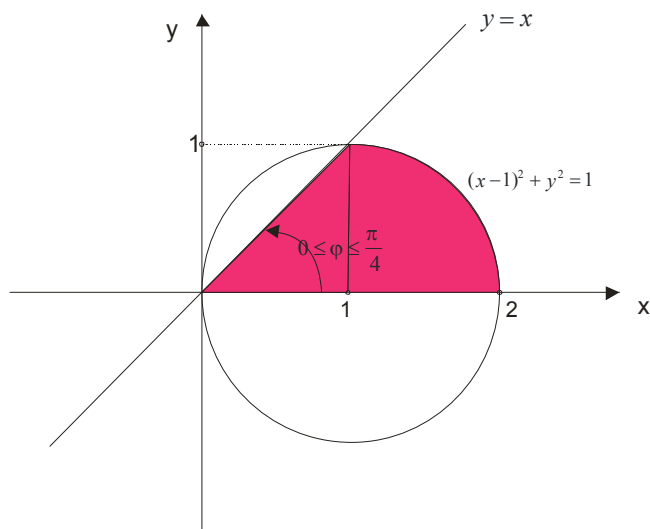
Spakujmo kružnicu i nacrtajmo sliku da vidimo o kojoj se oblasti radi...

$$x^2 + y^2 - 2x = 0$$

$$x^2 - 2x + 1 - 1 + y^2 = 0$$

$$(x-1)^2 + y^2 = 1$$

Preseci su očigledno u $x = 0$ i $x = 1$



$$x = r \cos \varphi$$

Uvodimo polarne koordinate: $y = r \sin \varphi$

$$|J| = r$$

$$x^2 + y^2 - 2x = 0$$

$$(r \cos \varphi)^2 + (r \sin \varphi)^2 - 2r \cos \varphi = 0$$

$$r^2(\cos^2 \varphi + \sin^2 \varphi) = 2r \cos \varphi$$

$$r^2 = 2r \cos \varphi \rightarrow r = 2 \cos \varphi$$

Odavde zaključujemo: $0 \leq r \leq 2 \cos \varphi$

Ugao ide od prave $y = 0$ do $y = x$, pa ugao ide od $0 \leq \varphi \leq \frac{\pi}{4}$

Sad možemo računati traženu površinu:

$$\begin{aligned} P &= \iint_D dx dy = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{2 \cos \varphi} r dr = \int_0^{\frac{\pi}{4}} \left(\frac{r^2}{2} \right) \Big|_0^{2 \cos \varphi} d\varphi = \int_0^{\frac{\pi}{4}} \left(\frac{4 \cos^2 \varphi}{2} \right) d\varphi = \\ &= 2 \int_0^{\frac{\pi}{4}} \cos^2 \varphi d\varphi \end{aligned}$$

Malo se pomognemo trigonometrijskim formulama: $\cos^2 \varphi = \frac{1 + \cos 2\varphi}{2}$

$$\begin{aligned} P &= \iint_D dx dy = 2 \int_0^{\frac{\pi}{4}} \cos^2 \varphi d\varphi = 2 \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2\varphi}{2} d\varphi = \int_0^{\frac{\pi}{4}} (1 + \cos 2\varphi) d\varphi = \\ &= \left(\varphi + \frac{1}{2} \sin 2\varphi \right) \Big|_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} + \frac{1}{2} \sin 2 \cdot \frac{\pi}{4} \right) - \left(0 + \frac{1}{2} \sin 2 \cdot 0 \right) = \boxed{\frac{\pi}{4} + \frac{1}{2}} \end{aligned}$$

Do sada smo upotrebljavali polarne i cilindrične koordinate.

Medjutim u ozbiljnijim zadacima moramo upotrebljavati takozvane **generalisane polarne koordinate** r i φ po formulama:

$$\left. \begin{aligned} x &= ar \cos^\alpha \varphi \\ y &= br \sin^\alpha \varphi \end{aligned} \right\} \rightarrow |J| = \alpha \cdot a b r \cdot \cos^{\alpha-1} \varphi \cdot \sin^{\alpha-1} \varphi$$

Vrednost za α se uzima u zavisnosti od konkretne situacije...

Gledamo da kod te date krive pogodnom vrednošću za α na levoj strani ostane samo r^2 . To je ideja.

Primer 4 .

Izračunati površinu ograničenu sa $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{h} + \frac{y}{k}$ ako su parametri a, b, h i k pozitivni.

Rešenje:

Najpre ćemo malo da prepakujemo zadatu krivu ...

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{h} + \frac{y}{k}$$

$$\frac{x^2}{a^2} - \frac{x}{h} + \frac{y^2}{b^2} - \frac{y}{k} = 0 \text{ dopunimo do punih kvadrata}$$

$$\left[\frac{x^2}{a^2} - \frac{x}{h} + \left(\frac{a}{2h} \right)^2 \right] - \left(\frac{a}{2h} \right)^2 + \left[\frac{y^2}{b^2} - \frac{y}{k} + \left(\frac{b}{2k} \right)^2 \right] - \left(\frac{b}{2k} \right)^2 = 0$$

$$\left(\frac{x}{a} - \frac{a}{2h} \right)^2 + \left(\frac{y}{b} - \frac{b}{2k} \right)^2 = \left(\frac{a}{2h} \right)^2 + \left(\frac{b}{2k} \right)^2$$

$$\left(\frac{x}{a} - \frac{a}{2h} \right)^2 + \left(\frac{y}{b} - \frac{b}{2k} \right)^2 = \frac{a^2}{4h^2} + \frac{b^2}{4k^2}$$

Sad razmišljamo. Zgodno bi bilo da uništimo ovo u zagradama. Zato ćemo uzeti da je :

$$\frac{x}{a} - \frac{a}{2h} = r \cos \varphi \rightarrow \frac{x}{a} = r \cos \varphi + \frac{a}{2h} \rightarrow \boxed{x = ar \cos \varphi + \frac{a^2}{2h}}$$

$$\frac{y}{b} - \frac{b}{2k} = r \sin \varphi \rightarrow \frac{y}{b} = r \sin \varphi + \frac{b}{2k} \rightarrow \boxed{y = br \sin \varphi + \frac{b^2}{2k}}$$

Dakle, uzimamo da je:

$$\left. \begin{array}{l} x = ar \cos \varphi + \frac{a^2}{2h} \\ y = br \sin \varphi + \frac{b^2}{2k} \end{array} \right\} \rightarrow |J| = abr$$

Sad da odredimo granice.

$$\left(\frac{x-a}{2h}\right)^2 + \left(\frac{y-b}{2k}\right)^2 = \frac{a^2}{4h^2} + \frac{b^2}{4k^2}$$

ovo sve daje r^2 jer smo tako izabrali

$$r^2 = \frac{a^2}{4h^2} + \frac{b^2}{4k^2}$$

$$r^2 = \frac{1}{4} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right) \rightarrow r = \frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}$$

Dobili smo granice za r : $0 \leq r \leq \frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}$

Ugao nema nikakvih “ograničenja”, pa je $0 \leq \varphi \leq 2\pi$

Sad računamo traženu površinu:

$$P = \iint_D dx dy = \int_0^{2\pi} d\varphi \int_0^{\frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}} ab r dr = ab \int_0^{2\pi} d\varphi \int_0^{\frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}} r dr$$

Kako se u integralu “po r ” uopšte i ne nalazi ugao φ odmah možemo napisati da je $\int_0^{2\pi} d\varphi = 2\pi$,

Dalje imamo:

$$P = \iint_D dx dy = \int_0^{2\pi} d\varphi \int_0^{\frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}} ab r dr = 2ab\pi \int_0^{\frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}} r dr =$$

$$= 2ab\pi \left(\frac{r^2}{2} \right) \Big|_0^{\frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}}} = ab\pi \left(\frac{1}{2} \sqrt{\frac{a^2}{h^2} + \frac{b^2}{k^2}} \right)^2 = ab\pi \frac{1}{4} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right) = \boxed{\frac{ab\pi}{4} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right)}$$

Primer 5.

Izračunati površinu ograničenu sa :

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 4$$

$$\frac{x}{a} = \frac{y}{b}$$

$$8\frac{x}{a} = \frac{y}{b}$$

$$x > 0, y > 0$$

Rešenje:

Upotrebićemo **generalisane polarne koordinate** : $\left. \begin{array}{l} x = ar \cos^\alpha \varphi \\ y = br \sin^\alpha \varphi \end{array} \right\} \rightarrow |J| = \alpha \cdot abr \cdot \cos^{\alpha-1} \varphi \cdot \sin^{\alpha-1} \varphi$ i

to:

$$\left. \begin{array}{l} x = ar \cos^3 \varphi \\ y = br \sin^3 \varphi \end{array} \right\} \rightarrow |J| = 3 \cdot abr \cdot \cos^{3-1} \varphi \cdot \sin^{3-1} \varphi \rightarrow \boxed{|J| = 3 \cdot abr \cdot \cos^2 \varphi \cdot \sin^2 \varphi}$$

Da vidimo sada granice i zašto smo baš izabrali da je $\alpha = 3$.

$\left. \begin{array}{l} x = ar \cos^3 \varphi \\ y = br \sin^3 \varphi \end{array} \right\}$ zamenimo u $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ i dobijamo:

$$\left(\frac{\cancel{\alpha}r \cos^3 \varphi}{\cancel{\alpha}}\right)^{\frac{2}{3}} + \left(\frac{\cancel{b}r \sin^3 \varphi}{\cancel{b}}\right)^{\frac{2}{3}} = 1$$

$$r^{\frac{2}{3}} \cos^2 \varphi + r^{\frac{2}{3}} \sin^2 \varphi = 1$$

$$r^{\frac{2}{3}} = 1 \rightarrow \boxed{r = 1}$$

Dalje $\left. \begin{array}{l} x = ar \cos^3 \varphi \\ y = br \sin^3 \varphi \end{array} \right\}$ zamenimo u $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 4$ i dobijamo:

$$\left(\frac{ar \cos^3 \varphi}{a}\right)^{\frac{2}{3}} + \left(\frac{br \sin^3 \varphi}{b}\right)^{\frac{2}{3}} = 4$$

$$r^{\frac{2}{3}} \cos^2 \varphi + r^{\frac{2}{3}} \sin^2 \varphi = 4$$

$$r^{\frac{2}{3}} = 4 \rightarrow \boxed{r = 8}$$

Dobili smo granice za r : $1 \leq r \leq 8$

Sad da odredimo granice za ugao:

$$\frac{x}{a} = \frac{y}{b}$$

$$\frac{ar \cos^3 \varphi}{a} = \frac{br \sin^3 \varphi}{b}$$

$$\cos^3 \varphi = \sin^3 \varphi \rightarrow \frac{\sin^3 \varphi}{\cos^3 \varphi} = 1 \rightarrow \operatorname{tg}^3 \varphi = 1 \rightarrow \boxed{\varphi = \operatorname{arctg} 1}$$

I još imamo:

$$8 \frac{x}{a} = \frac{y}{b}$$

$$8 \frac{ar \cos^3 \varphi}{a} = \frac{br \sin^3 \varphi}{b}$$

$$8 \cos^3 \varphi = \sin^3 \varphi \rightarrow \frac{\sin^3 \varphi}{\cos^3 \varphi} = 8 \rightarrow \operatorname{tg}^3 \varphi = 2^3 \rightarrow \boxed{\varphi = \operatorname{arctg} 2}$$

Dakle: $\operatorname{arctg} 1 \leq \varphi \leq \operatorname{arctg} 2$

Sad računamo traženu površinu:

$$P = \iint_D dx dy = \int_{\operatorname{arctg} 1}^{\operatorname{arctg} 2} d\varphi \int_1^8 3 \cdot ab r \cdot \cos^2 \varphi \cdot \sin^2 \varphi dr = 3ab \int_{\operatorname{arctg} 1}^{\operatorname{arctg} 2} \cos^2 \varphi \cdot \sin^2 \varphi d\varphi \int_1^8 r dr =$$

$$\text{Kako je } \int_1^8 r dr = \frac{r^2}{2} \Big|_1^8 = \frac{64}{2} - \frac{1}{2} = \frac{63}{2}, \text{ imamo}$$

$$= \frac{63}{2} \cdot 3ab \int_{\operatorname{arctg} 1}^{\operatorname{arctg} 2} \cos^2 \varphi \cdot \sin^2 \varphi d\varphi$$

Ovaj integral ćemo najlakše rešiti ako spakujemo podintegralnu funkciju koristeći formule iz trigonometrije:

$$\cos^2 \varphi \cdot \sin^2 \varphi = \frac{4}{4} \cos^2 \varphi \cdot \sin^2 \varphi = \frac{\sin^2 2\varphi}{4} = \frac{1}{4} \sin^2 2\varphi = \frac{1}{4} \left(\frac{1 - \cos 4\varphi}{2} \right) = \frac{1}{8} (1 - \cos 4\varphi)$$

Sad imamo:

$$P = \frac{63}{2} \cdot 3ab \int_{\arctg 1}^{\arctg 2} \cos^2 \varphi \cdot \sin^2 \varphi d\varphi = \frac{189}{16} ab \int_{\arctg 1}^{\arctg 2} (1 - \cos 4\varphi) d\varphi$$

Zamenimo granice, spakujemo malo rešenje i dobijamo:

$$P = \frac{189}{16} ab \cdot \left(\arctg \frac{1}{3} + \frac{6}{25} \right)$$

Primer 6 .

Izračunati površinu ograničenu sa :

$$x^2 = ay$$

$$x^2 = by$$

$$x^3 = cy^2$$

$$x^3 = dy^2$$

$$(0 < a < b) \wedge (0 < c < d)$$

Rešenje:

Ovde je zgodno uzeti smene u i v .

Ali kako birati?

Pogledajmo prve dve jednačine:

$$x^2 = ay \rightarrow \frac{x^2}{y} = a$$

$$x^2 = by \rightarrow \frac{x^2}{y} = b$$

Uzećemo da je $u = \frac{x^2}{y}$

Iz preostale dve imamo:

$$x^3 = cy^2 \rightarrow \frac{x^3}{y^2} = c$$

$$x^3 = dy^2 \rightarrow \frac{x^3}{y^2} = d$$

Zgodno je uzeti: $\boxed{v = \frac{x^3}{y^2}}$

Dakle, uvodimo smene:

$$u = \frac{x^2}{y}$$

$$v = \frac{x^3}{y^2}$$

Odavde moramo izraziti x i y:

$$u = \frac{x^2}{y} \rightarrow y = \frac{x^2}{u} \text{ (ovo zamenimo u drugu jednačinu)}$$

$$v = \frac{x^3}{y^2} \rightarrow x^3 = y^2 v \rightarrow x^3 = \left(\frac{x^2}{u}\right)^2 \cdot v \rightarrow x^3 = \frac{x^4}{u^2} v \rightarrow \boxed{x = \frac{u^2}{v}}$$

$$y = \frac{x^2}{u} \rightarrow y = \frac{\left(\frac{u^2}{v}\right)^2}{u} \rightarrow \boxed{y = \frac{u^3}{v^2}}$$

Tražimo Jakobijan:

$$\left| \frac{D(x, y)}{D(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ \frac{3u^2}{v^2} & -\frac{2u^3}{v^3} \end{vmatrix} = \left| -\frac{4u^4}{v^4} + \frac{3u^4}{v^4} \right| = \left| -\frac{u^4}{v^4} \right| = \frac{u^4}{v^4}$$

Da odredimo granice:

$$\left. \begin{array}{l} u = \frac{x^2}{y} = a \\ u = \frac{x^2}{y} = b \end{array} \right\} \rightarrow \boxed{a \leq u \leq b}$$

$$\left. \begin{array}{l} v = \frac{x^3}{y^2} = c \\ v = \frac{x^3}{y^2} = d \end{array} \right\} \rightarrow \boxed{c \leq v \leq d}$$

Sad možemo izračunati površinu:

$$P = \iint_D dx dy = \int_a^b du \int_c^d \frac{u^4}{v^4} dv = \int_a^b u^4 du \int_c^d \frac{1}{v^4} dv$$

Ova dva integrala nije teško rešiti i dobijamo:

$$P = \frac{1}{15} (b^5 - a^5) \left(\frac{1}{c^3} - \frac{1}{d^3} \right)$$