there is $a \in A$ such that $ax \neq 0$. $Ra \cap x^l \neq (0)$ since x^l is large. Hence there is $r_1 a \neq 0$ in Ra such that $(r_1 a)x = 0$. This is impossible since $(r_1 a)^r = a^r$.

THEOREM 3.2. *Let $R \in C$ and $L_r^*(R)$ be atomic. If R has a regular ring Q as a two-sided quotient ring* (in the sense of R. E. Johnson) *then $L_l^*(R)$ is also atomic. If, in addition, some uniform right ideal of R is a prime ring, then Q is a division ring and R is a right and left Ore-domain.*

Proof. If Q is a regular ring then each atom of $L_r^*(Q)$ is a minimal right ideal by Lemma 2.3. Let $\overline{A}$ be an atom of $L_r^*(Q)$. Then $\overline{A} = eQ$ for some idempotent e in Q. Since $\overline{A}$ is a minimal right ideal, Qe is a minimal left ideal. Thus $Qe \cap R$ is an atom of $L^*(R)$. In case atom $A \in L_r^*(R)$ is a prime ring then $(\overline{A})^l = (0)$ by Lemma 2.2. Hence e must be equal to 1. Thus $\overline{A} = Q$ and Q is a division ring.

REMARK. The Theorem 3.2 asserts that in general the maximal two-sided quotient ring of a ring R with $R_r^\Delta = (0)$ and $R_l^\Delta = (0)$ is not necessarily regular.

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References

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A TYPE OF "GAMBLER'S RUIN" PROBLEM

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In his book "Tomorrow's Math" ([1], page 114) Ogilvy quotes the following as an unsolved problem: "Three men have respectively l, m and n coins which they match so that the odd man wins. Whenever all coins appear alike they repeat the throw. Find the average number of tosses required until one man is forced out of the game." In this paper we derive the solution to this problem, which turns out to be unexpectedly simple.

Let $E(l, m, n)$ be the expected number of tosses. Of course we assume that the coins are "fair". If one toss is made then there is

(i) probability $\frac{1}{4}$ that the coins appear alike, in which case $E(l, m, n)$ further tosses are expected.

(ii) probability $\frac{1}{4}$ that the first player wins, in which case $E(l+2, m-1, n-1)$ further tosses are expected.

(iii) and (iv) probabilities as for (ii) but for the second or third players.

From these remarks we see that

$$\begin{aligned} E(l, m, n) = {} & \tfrac{1}{4}[1 + E(l, m, n)] + \tfrac{1}{4}[1 + E(l + 2, m - 1, n - 1)] \\ & + \tfrac{1}{4}[1 + E(l - 1, m + 2, n - 1)] + \tfrac{1}{4}[1 + E(l - 1, m - 1, n + 2)] \end{aligned}$$

or

$$\tfrac{3}{4}E(l, m, n) = 1 + \tfrac{1}{4}E(l+2, m-1, n-1) + \tfrac{1}{4}E(l-1, m+2, n-1) + \tfrac{1}{4}E(l-1, m-1, n+2).$$

It will be convenient to put $l=p+1$, $m=q+1$, $n=r+1$ and thus obtain

$$E(p+1, q+1, r+1) = \tfrac{4}{3} + \tfrac{1}{3}E(p+3, q, r) + \tfrac{1}{3}E(p, q+3, r) + \tfrac{1}{3}E(p, q, r+3). \tag{1}$$

Throughout the game the sum $l+m+n=p+q+r+3$, the total number of coins, remains the same. Therefore although equation (1) has the appearance of a recursion formula, it cannot be used directly to calculate $E(l, m, n)$ in terms of E's with smaller values of l, m and n, as one would expect from a recursion formula. For example, if $l+m+n=7$ then, since $E(l, m, n)=0$ if any one of l, m or n is zero, we can derive from (1) the following equations:

$$E(5, 1, 1) = E(1, 5, 1) = E(1, 1, 5) = \tfrac{4}{3}$$

$$E(4, 2, 1) = \tfrac{4}{3} + \tfrac{1}{3}E(3, 1, 3)$$

$$E(3, 3, 1) = \tfrac{4}{3} + \tfrac{1}{3}E(2, 2, 3)$$

$$E(3, 2, 2) = \tfrac{4}{3} + \tfrac{1}{3}E(5, 1, 1) + \tfrac{1}{3}E(2, 4, 1) + \tfrac{1}{3}E(2, 1, 4).$$

These three equations can be solved and we find, since the function E is clearly symmetrical in its arguments, that

$$E(4, 2, 1) = E(2, 4, 1) = E(2, 1, 4) = \tfrac{32}{15}$$

$$E(3, 3, 1) = E(3, 1, 3) = E(1, 3, 3) = \tfrac{12}{5}$$

and

$$E(2, 2, 3) = E(2, 3, 2) = E(3, 2, 2) = \tfrac{16}{5} \cdot$$

This method of calculating the values of $E(l, m, n)$ will clearly become increasingly tedious as the value of $l+m+n$ increases, and in any case is not likely to yield a general formula. Moreover the usual techniques of using counting series or generating functions do not seem to be of much help in a problem of this kind.

Let us turn to a simpler problem. Two men match coins. If one is "heads" and the other "tails," then "heads" wins. If the coins are alike, neither player wins. As before, we must find the expected number of tosses before one player is ruined. It is easily seen that this problem is equivalent to a random walk on a line, with probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ of moving one unit to the left, staying put, or moving one unit to the right. The equation analogous to (1) is

$$E(p+1, q+1) = 2 + \tfrac{1}{2}E(p+2, q) + \tfrac{1}{2}E(p, q+2) \tag{2}$$

with an obvious notation. If we write $u_l=E(l, N-l)$, where N is the total number of coins, then (2) becomes

$$u_{p+2} - 2u_{p+1} + u_p = -4. \tag{3}$$

This is a straightforward difference equation, the solution of which, by the usual methods is

$$u_p = Cp + D - 2p(p-1),$$

where C and D are constants. We require u_p to be zero when $p=0$ and when $p=N$. For this we must have $D=0$ and $C=2(N-1)$, and we obtain

$$E(l, m) = 2(N-1)l - 2l(l-1) = 2lm.$$

This is an extremely simple result, and it would be sanguine in the extreme to expect a similar result to hold for equation (1), but there is no harm in trying. Let us assume $E(l, m, n) = Klmn$, where K is a constant. Equation (1) becomes

$$K(p+1)(q+1)(r+1) = \tfrac{4}{3} + \tfrac{1}{3}K\{(p+3)qr + p(q+3)r + pq(r+3)\} \tag{4}$$

or $K(p+q+r)+K=\frac{4}{3}$, since the terms of the second and third degrees cancel. Hence

$$K = \frac{4}{3(p+q+r+1)} = \frac{4}{3(l+m+n-2)}, \text{ a constant!}$$

Thus we have $E(l, m, n) = (4lmn)/3(l+m+n-2)$, which furnishes one solution of the problem much more easily than one would have any reason to expect.

A similar problem with four players would be unlikely to have a simple solution. In the equation analogous to (4), in fact, the terms of degree 4 would cancel, those of degree 3 might do so, but those of degree 2 would almost certainly not. Hence no constant K would cause the equation to be satisfied. A typical problem of this type would be the following. Four players match coins, the odd man winning and collecting a coin from each of the other three. If all coins are alike, or if there are two heads and two tails, then no coins change hands. Given the number of coins each player has in the beginning, find the expected number of tosses before one player loses all his coins. Other problems with four players will result from different rulings concerning the way the coins change hands.

It is readily shown that the function $Kl_1l_2l_3l_4$, where l_1, l_2, l_3, l_4 are the numbers of coins, is *not* the solution to this problem. Other plausible possibilities, such as a linear combination of the symmetric functions $l_1l_2l_3l_4$, $\Sigma l_1l_2l_3$, Σl_1l_2 and Σl_1 can also be easily ruled out. It looks as though this problem is one that does not have an easy solution.

Reference

1. C. Stanley Ogilvy. Tomorrow's Math; unsolved problems for the amateur, Oxford University Press, New York, 1963.

Editorial Note. This is Problem 4003 in this MONTHLY, 48 (1941) 483, proposed by G. W. Petrie. The galley proofs of the present issue contained a solution by F. Göbel (Netherlands).