An Intriguing Geometry Problem

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1   History and Background

Problem. Let ABC be an isosceles triangle (AB=AC) with $\angle BAC = 20°$. Point D is on side AC such that $\angle DBC = 60°$. Point E is on side AB such that $\angle ECB = 50°$. Find, with proof, the measure of $\angle EDB$.

This geometry problem which is the major focus of the talk today dates back to at least 1922, when it appeared in the *Mathematical Gazette*, Volume 11, p. 173. It appears to be an easy problem, but it is deceivingly difficult. I first saw the problem in the late 1960's when my go teacher at the time gave it to me. I worked for a long time on the problem. However, I persisted in treating it as an elementary angle-chasing problem and did not solve it. A few years later the problem surfaced in a talk for high school students by Bill Leonard from California State College in Fullerton. As soon as he showed the problem on the screen, I covered my ears and put my head down, because I did not want to hear the solution. Nevertheless, I did not manage to solve the problem and finally when I encountered it in *Geometry Revisited* [1] by Coxeter and Greitzer, I read the solution one line at a time, trying to complete the proof. It was still difficult for me, since I was teaching Junior High School at the time and had not studied any geometry since high school. I then found the problem in some books I ordered for my classes: *Trigonometric Novelties* [10] and *One Hundred Mathematical Curiosities* [9]. Both were written by William Ransom in the 1950's. I had not even considered trying to prove it using trigonometry. Of course there were no calculators in the early 1970's selling for less than $200 that had trigonometric function keys . The Mathematical Association of America was now publishing a new series of books, the Dolciani Mathematical Expositions. The second volume, *Mathematical Gems II* [3] by Ross Honsberger, had a section entitled "Four Minor Gems from Geometry". In this section, a 1951 proof by S.T. Thompson was presented based on intersecting diagonals of a regular 18-gon. Also during this period, a problem with 60° and 70° angles instead of 60° and 50° angles surfaced in the 1976 Carleton University Mathematics Competition for high school students. It was widely discussed in *Crux Mathematicorum* [2] with a call for non-trigonometric solutions. Many were forthcoming in the months that followed. In the May-June 1994 issue of *Quantum* there appeared a very interesting article entitled " Nine Solutions to One Problem" [5] by Constantine Knop. This talk will be a discussion of these solutions. In 1997, an article in *Mathematical Horizons* [7] contained an article entitled "A Better Angle From Outside" by Andy Liu which discusses several problems that can be solved with the key idea used in the sixth proof in Knop's article. In 2000, *Essays on Numbers and Figures* [8] by V.V. Prasolov became Volume 16 in the American Mathematical Society series *Mathematical World*. The essay in this volume, "Intersection Points of the Diagonals

of Regular Polygons", was to be the main topic of the talk today, but there is too much else to discuss, so it will only be mentioned as a generalization of the method of S.T. Thompson, mentioned above. Last year, *Mathematical Chestnuts from Around the World* [4] by Ross Honsberger, was published as Dolciani Mathematical Exposition Number 24. A problem from Knop's article, proposed by a gifted ninth grader, is discussed and three solutions are given. (See problem 1 below) Also last year, an article entitled "Dividable Triangles— What Are They?" came out in the May issue of *Mathematics Teacher* [6] which approaches these problems from the different point of view of dissecting isosceles triangles into isosceles triangles.

2 Eight Solutions

Here are the starting points and sketches for eight solutions.

1. Draw segment DF parallel to BC with F on AB. Draw CF intersecting BD at G. Now find the equilateral triangles and isosceles triangles.

2. Use the Law of Sines in triangle BED and triangle BCD. Use $BE = BC$ to connect the results. Simplify and solve for $\angle EDB$.

3. Draw lines through D and B parallel to BC and DC, respectively, intersecting at H. Draw CG with G on BD and $\angle GCB = 60°$. Show E is the incenter of triangle BDH.

4. Mark K on AC such that $\angle KBC = 20°$. Draw KB and KE. Show $BE = BC = BK = KE = KD$.

5. (Maria Gelband) Reflect E through AC to point H. Show D is on the circumcircle of triangle BEH.

6. (Sergei Saprikin) Let the bisector of $\angle ABC$ intersect AC at point T. Show D is an excenter of triangle BET.

7. (Alexey Borodin) Let O be the circumcenter of triangle DEC. Show BD is the perpendicular bisector of EO.

8. (Alexander Kornienko) Reflect triangle ABC through AB to triangle ABC' and also relect it through AC to triangle ACB'. Show that C', E, and D are collinear.

3 Problems

1. (Sergey Yurin, 9th grade) In an isosceles triangle ABC, $AB = AC$, and $\angle A = 20°$. Point P is taken on the side AC such that $AP = BC$. Find $\angle PBC$.

   Use the idea of excenters and incenters to solve the following problems.

2. (Carleton University Mathematics Competition for High School Students, 1976) ABC is an isosceles triangle with $\angle ABC = \angle ACB = 80°$. P is the point on AB such that $\angle PCB = 70°$. Q is the point on AC such that $\angle QBC = 60°$. Find $\angle PQA$.

3. (Pythagoras Olympiad in The Netherlands, 1980) In triangle ABC, point D is such that $\angle DCA = \angle DCB = \angle DBC = 10°$ and $\angle DBA = 20°$. Find the measure of $\angle CAD$.

2

4. (Alberta High School Mathematics Competition, 1989–90) In quadrilateral $ABCD$ with diagonals BD and AC, $\angle ABD = 40°$, $\angle CBD = 70°$, $\angle CDB = 50°$, $\angle ADB = 80°$. Find the measure of $\angle CAD$.

5. (Junior Problem A-6, Tournament of Towns, Spring 1997) Let P be a point inside triangle ABC with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$ and $\angle ACP = 30°$. Find the measure of $\angle BPC$.

6. Senior Problem A-2, Tournament of Towns, Spring 1997) D is the point on BC and E is the point on CA such that AD and BE are the bisectors of $\angle A$ and $\angle B$ of triangle ABC. If DE is the bisector of $\angle ADC$, find the measure of $\angle A$.

4 References

1. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, Mathematical Association of America, 1967.

2. Crux Mathematicorum, Problem 134, 2(1976) p. 68.

3. Ross Honsberger, "Four Minor Gems from Geometry", *Mathematical Gems II*, Mathematical Association of America, 1976.

4. Ross Honsberger, "Three Solutions to a Variation on an Old Chestnut", *Mathematical Chestnuts from Around the World*, Mathematical Association of America, 2001.

5. Constantine Knop, " Nine Solutions to One Problem", *Quantum*, May-June 1994, pp. 46–49.

6. Roza Leikin, "Dividable Triangles—What Are They?", *Mathematics Teacher*, May 2001, pp. 392–398.

7. Andy Liu, "A Better Angle From Outside", *Mathematical Horizons*, November 1997, pp. 27–29.

8. V.V. Prasolov, *Essays on Numbers and Figures*, American Mathematical Association, 2000.

9. William Ransom, *One Hundred Mathematical Curiosities*, J. Weston Walsh, 1955.

10. William Ransom, *Trigonometric Novelties*, J. Weston Walsh, 1959.

11. P.J. Taylor and A.M. Storozhev, *Tournament of Towns 1993–1997*, Australian Mathematics Trust, 1998.

If you have comments, questions or find glaring errors, please contact me by e-mail at the following address: trike@ousd.k12.ca.us

5   Eight Solutions (more detailed)

1. Draw segment DF parallel to BC with F on AB. Draw CF intersecting BD at G. Triangles BGC and DGF are equilateral. Triangle CEB is isosceles ($\angle BEC = 50 = \angle BCE$), so $BE = BC = BG$. Then triangle BEG is isosceles with a vertex angle of $20°$ and base angles of $80°$. Since $80° + \angle EGF + 60° = 180°$, we have $\angle EGF = 40°$. But in triangle BFC, $\angle BFC = 40°$, so triangle EFG is isosceles and $EFGD$ is a kite.

2. Use the Law of Sines in triangle BED and triangle BCD and $BE = BC$ to get $\dfrac{\sin(160° - x)}{\sin x} = \dfrac{BD}{BE} = \dfrac{BD}{BC} = \dfrac{\sin(80°)}{\sin 40°}$. Simplify to get $\sin(20° + x) = 2\cos 40° \sin x$.

3. Draw lines through D and H parallel to BC and DC, respectively, intersecting at H. Draw CG with G on BD and $\angle GCB = 60°$. $BE = BC = CG$, $BH = CD$ and $\angle EBH = \angle GCD = 20°$, so $\triangle EBH \cong \triangle GCD$. Therefore $\angle BHE = \angle CDG = 40°$. But $\angle BHD = 80°$, so $\angle EHD = 40° = \angle BHE$ and E is the incenter of triangle BDH.

4. Mark K on AC such that $\angle KBC = 20°$. Draw KB and KE. $BE = BC = BK$ and $\angle EBK = 60°$, so triangle EBK is equilateral and triangle KBC is isosceles with $\angle BKC = 80°$. $\angle EKD = 40°$ since $\angle EKC = 140°$. In triangle BDK, $\angle BDK = 40°$, so that triangle BKD is isosceles with $KD = KB = KE$. Then triangle KDE is isosceles with a $40°$ vertex angle at K. The base angles are $70°$, so $\angle EDC = 30°$.

5. Reflect E through AC to point H. Triangle ECH is equilateral with line CD, the perpendicular bisector of EH. But $BE = BC$, so $BCHE$ is a kite. Draw symmetry line BH. Ray BD bisects angle EBH. Consider the circumcircle of triangle BEH. Line BD passes through the midpoint of arc EH and ray BD passes through the midpoint of arc EH. Since D is on the ray and the line it must be on the circle. Then the measure of inscribed angle EDB has the same measure as inscribed angle EHB

6. Let the bisector of $\angle ABC$ intersect AC at point T. BD bisects $\angle EBT$. $\triangle BET \cong \triangle BCT$ by SAS, so $\angle ETB = \angle CTB = 60°$. Therefore $\angle ETD = 60°$ which is the measure of the angle formed by the extension of BT and AT and D is an excenter of triangle BET. (D is equidistant from the lines BT, ET, and BE.) Therefore ED bisects $\angle AET$ and we have $\angle BED = 50° + 30° + 50° = 130°$.

7. Let O be the circumcenter of triangle DEC. Then central angle EOD is twice the measure of inscribed angle ECD. So $\angle EOD = 60°$ and triangle EOD is equilateral. Now D lies on the perpendicular bisector of EO and BD bisects angle BEO. If the perpendicular bisector of EO is different from BD, then D would lie on the circumcircle of triangle EOB. Then opposite angles of cyclic quadrilateral $EBOD$ would be supplementary, but $\angle EBO + \angle EDO = 40° + 60° \neq 180°$. Therefore BD bisects $\angle EDO$.

8. Reflect triangle ABC through AB to triangle ABC' and also reflect it through AC to triangle ACB'. $AC'B'$ is equilateral and $\angle BC'B' = 20°$. Draw $C'E$. Therefore $C'E$ bisects $\angle AC'B'$. Draw DB. $AD = DB = DB'$, so D is on the perpendicular bisector of AB' which coincides with the angle bisector since the triangle is equilateral.