ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4425. *Proposed by P. A. Piza, San Juan, Puerto Rico*

There are infinitely many triangular numbers (numbers of the form $k(k+1)/2$) which are also perfect squares, viz.

$$[(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2]/32.$$

Find numbers which are at the same time a sum of two squares and a sum of two triangular numbers.

4426. *Proposed by D. J. Newman, New York University*

Let a, b, c, d all have modulus unity. Prove that in the unit circle the polynomial

$$P(z) = az^3 + bz^2 + cz + d$$

has a maximum modulus not less than $\sqrt{6}$. What is the result for a polynomial of degree greater than three?

4427. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let

$$f_n(x) = \prod_{i=1}^{n} (x - x_i), \qquad -1 \leqq x_i \leqq 1.$$

Prove that there cannot exist numbers a, b such that

$$|f_n(a)| \geqq 1, \qquad |f_n(b)| \geqq 1, \qquad -1 < a < 0 < b < 1.$$

SOLUTIONS

Square Inscribed in Arbitrary Simple Closed Curve

4325 [1949, 39]. *Proposed by Orrin Frink, Pennsylvania State College.*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

Note by A. M. Gleason, Cambridge, Massachusetts. I would like to point out the inadequacy of the proof printed in the June issue [1950, 423] as a solution of problem 4325. First, it is stated that it is possible to deform a horizontal chord in such a way that it is always horizontal and moves from the first contact to the last contact. While this is easy to see when the curve has only a finite number of maxima and minima, it certainly requires proof that it can be done in general.

Second, the perpendicular bisector of the moving horizontal chord is alleged to have continuous behavior. Since the perpendicular bisector of the curve may have many different intersections with the curve it is easy to give examples in which, say, the first intersection counting from below is not continuous. It may be possible to get around this difficulty by the device of retrograding as before, but this must be combined with the demand for a retrograde motion for XX' and proof is certainly required.

Third, it is assumed that the rhombi constructed for each direction depend continuously on the direction. This is a difficulty which cannot be circumvented by the devices previously employed. Let A, B, C, D be the vertices of a rhombus. A simple closed curve may be easily constructed containing A, B, C, D but not containing any other rhombus near $ABCD$. To do this, simply let A and C lie on cusps pointing to the right while B and D lie on cusps pointing to the left.

Note by J. J. Schaeffer, Instituto de Matematica y Estadistica, Montevideo, Uruguay. In the second part of the proof the greatest difficulties are encountered. This applies particularly to the statement "then rotate m_x continuously through 90° until it reaches the position m_y; now $XX' < YY'$." There seems to be no reason to assume the possibility of returning by continuous rotation to the initial position of the rhombus, and even in the case of fairly simple curves the proof encounters apparently insurmountable obstacles.

For convex figures, however, the idea can be used, as it can be easily proved that convex figures have the property that for any slope there is essentially only one rhombus with a diagonal parallel to this slope. This is the same idea as Emch's, but in order to apply it to general convex curves, which may contain rectilinear segments or angles, the use of continuity should be avoided, and replaced by considerations of limit points of open sets.

Similar comments were received from Arthur Rosenthal, and G. A. Dirac. Further information and discussion of this interesting problem will be welcomed.

A Prime Representing Function

4337 [1949, 186]. *Proposed unsigned.*

If the numbers R_{nk} are defined by

$$\frac{1 - z^2}{\sin \pi z} \prod_{k=2}^{n} \sin (\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk} z^k,$$