It should be remarked that the series in this problem completes a set of well known formulas:
Write $[0]=1$, $[n]=(1-x)(1-x^2)\cdots(1-x^n)$; then

$$(i)\qquad \sum_{n\geqq 0}\frac{x^n}{[n]}=\prod_{k\geqq 1}\frac{1}{1-x^k},$$

$$(ii)\qquad \sum_{n\geqq 0}\frac{x^{n^2}}{[n]^2}=\prod_{k\geqq 1}\frac{1}{1-x^k},$$

$$(iii)\qquad \sum_{n\geqq 0}\frac{x^{n^2}}{[n]}=\prod_{k\geqq 1}\frac{1}{(1-x^{5k-4})(1-x^{5k-1})},$$

$$(iv)\qquad \sum_{n\geqq 0}\frac{x^n}{[n]^2}=\prod_{k\geqq 1}\frac{1}{(1-x^k)^2}\sum_{m\geqq 0}(-1)^m x^{(m^2+m)/2}.$$

The first two are very well known, they probably go back to Euler; (*iii*) is one of the famous Rogers-Ramanujan identities; while (*iv*) is the present problem.

In much the same way as (*iv*) we may derive

$$(v)\qquad \sum_{n\geqq 0}\frac{x^{2n}}{[2n]}=\prod_{k\geqq 1}\frac{1}{1-x^k}\sum_{n\geqq 0}(-1)^n x^{n^2}$$

which has an interesting interpretation in terms of partition functions. If $p(m)$ is the unrestricted partition function, the right member of (*v*) can be written

$$\sum_{N\geqq 0} x^N\sum_{n\geqq 0}(-1)^n p(N-n^2).$$

On the other hand the left member of (*v*) is easily seen to generate the number of partitions of an integer into parts the largest of which is even, or, by reading the conjugate graph, into an even number of parts. Denoting this partition function by $p_E(N)$, we have proved

$$(vi)\qquad p_E(N)=p(N)-p(N-1)+p(N-4)-p(N-9)+\cdots.$$

It would be interesting to see if there is a combinatorial proof of (*vi*).

Square Inscribed in Arbitrary Simple Closed Curve

4325 [1949, 39]. *Proposed by Orrin Frink, Pennsylvania State College*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

Solution by C. S. Ogilvy, Trinity College, Hartford. Let m_{x_i}, or more simply, m_x, be a system of horizontal straight lines in the plane, some of which cut the curve at X_i and X_i', or more simply X and X'. Let m_{x_1} avoid the curve by passing wholly below it, and m_{x_2} avoid the curve by passing wholly above it (Jordan.) If m_x is now moved continuously from position x_1 to position x_2, it will pass across the curve, producing two continuous sequences of points of intersection

X and X'. In the event of concavities, choose those intersections which allow X and X' to move continuously along the curve. Thus there will be times when the motion of m_x and, say, X will be retrograde while accommodating the wanderings of X'; but at all times the motion of all three, and in particular the length XX', will be continuous functions.

Now let YY' be the perpendicular bisector of XX', meeting XX' in P and terminated by its intersections with the curve. There will be an initial (first contact) position of m_x where $Y'P > YP = 0$, and a final (last contact) position where $0 = Y'P < YP$. But because of the continuous behavior of XX', YY' of m_y moves continuously also, and there must be some position of YY' where $YP = Y'P$ (Bolzano). This means that there is always an inscribed rhombus whose diagonals are horizontal and vertical. It follows that there is an inscribed rhombus for any slope of the diagonals (replace the word horizontal by parallel in the first sentence).

To continue: either $XX' = YY'$, in which case the rhombus is a square and the theorem is proved; or $XX' \neq YY'$. Suppose $XX' > YY'$. Then rotate m_x continuously through 90° until it reaches the position m_y; now $XX' < YY'$. Therefore there must have been an intermediate position of the rhombus (Bolzano again) where $XX' = YY'$. This is the position of the required square.

Editorial Note. V. L. Klee refers to a paper (in Russian) by Snirelman, L. G., *On certain geometrical properties of closed curves*, Uspehi Matem. Nauk 10, 34–44 (1944). See also abstract in Math. Rev., 7, 35 (1946). The case where the curve is convex is given by Emch, *Some properties of closed convex curves in a plane*, American Journal of Mathematics, vol. 35, 1913, pp. 407–412.

Function with Natural Boundary

4329 [1949, 40]. *Proposed by A. W. Goodman, University of Kentucky*

Let θ be an irrational number, $a = e^{i\theta\pi}$. Prove that

$$f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$$

has the unit circle as a natural boundary.

Solution by R. C. Buck, Brown University. The function $f(z)$ obeys the equation

$$f(z) - 1 = azf(a^2 z). \tag{1}$$

$f(z)$ has at least one singularity, z_0, on the unit circle. From (1) $a^2 z_0$ is also a singularity; iterating this, $a^{2m} z_0$ is a singularity for $m = 0, 1, \cdots$. Since θ is irrational, these points are everywhere dense on the unit circle, which is consequently a natural boundary for $f(z)$. If θ is rational, then $f(z)$ is a rational function.

Also solved by George Piranian.