On Lambert's proof of the irrationality of π

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The irrationality of π was first proved by J. H. Lambert in 1761 in his paper [**3**] (reprinted in [**4**, pp. 112–159]). Lambert's argument is the following. First he proves the formula

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{\ddots}}}} \tag{1}$$

Then Lambert shows, by an argument of infinite descent, that if $x \neq 0$ is rational then the right hand side of (1) is irrational. Since $\tan(\pi/4) = 1$ is rational, this implies that π is irrational.

Lambert's proof is seldom reproduced in books on number theory. The reason is clear: a rigorous proof of (1) cannot avoid questions of convergence of continued fractions, and if our aim is just to prove the irrationality of π then this digression is not worth the effort. (The last monograph that gives Lambert's argument in detail seems to be Chrystal's Algebra [**1**].) The "usual" proofs avoid continued fractions and use variants of Hermite's idea: if π were rational then certain sums or integrals would be integers, contradicting estimates showing that the actual values lie in $(0, 1)$; see Niven's book [**5**]. In the notes on Chapter 2, Niven also gives a list of papers following this line. J. Popken published several papers on the subject that contain variants of Hermite's argument [**6**], [**7**], [**8**]. The paper [**9**] is different; here Popken reproduces Lambert's computation and infers, in a particularly simple way, Lambert's theorem: if $x \neq 0$ is rational then $\tan x$ is irrational.

In this paper we further simplify [**9**] by replacing its computations with Gauss' functional equation. This gives a very simple proof of the irrationality of $\tan x$ (and also of $f(x)$ for a wide class of other functions) whenever $x \neq 0$ is rational. The irrationality of π follows. We also give a self-contained proof of (1) using the same device.

Lambert's original computation leading to (1) was somewhat tedious; he divided the power series of $\sin x$ by that of $\cos x$ using a version of Euclid's algorithm, and determined the quotients and the remainders. This computation was simplified by Gauss [**2**], who determined the continued fraction expansions of the hypergeometric series using their functional equations. If we want to prove only (1), then, following Gauss' argument, we may restrict our attention to the one-parameter family

$$f_k(x) = 1 - \frac{x^2}{k} + \frac{x^4}{k(k+1)\cdot 2!} - \frac{x^6}{k(k+1)(k+2)\cdot 3!} + \cdots.$$

It is easy to see that the series defining f_k converges for every x and for every

$k \neq 0, -1, -2, \ldots$. A simple computation shows that

$$\text{if } k = 1/2 \text{ then } k(k+1) \cdots (k+n-1) \cdot n! = (2n)!/4^n,$$

and

$$\text{if } k = 3/2 \text{ then } k(k+1) \cdots (k+n-1) \cdot n! = (2n+1)!/4^n.$$

Therefore we have

$$f_{1/2}(x) = \cos(2x) \text{ and } f_{3/2}(x) = \frac{\sin(2x)}{2x}$$

for every x. It is also easy to check, by comparing the coefficients of x^{2n}, that

$$\frac{x^2}{k(k+1)} f_{k+2}(x) = f_{k+1}(x) - f_k(x) \tag{2}$$

for every x and for every $k \neq 0, -1, -2, \ldots$. In the proof of the following theorem we combine (2) with the argument of [**9**].

Theorem 1. *If $x \neq 0$ and x^2 is rational, then $f_k(x) \neq 0$ and $f_{k+1}(x)/f_k(x)$ is irrational for every $k \in \mathbf{Q}$, $k \neq 0, -1, -2, \ldots$.*

Proof: First we show that

$$\lim_{r \to \infty} f_r(x) = 1. \tag{3}$$

Indeed, since $x^{2n}/n! \to 0$ as $n \to \infty$, there is some $K > 0$ such that $|x^{2n}/n!| \leq K$ for every n. Therefore, if $r > 1$, then $|f_r(x) - 1| \leq \sum_{n=1}^{\infty} K/r^n = K/(r-1)$, from which (3) follows.

Let x be a nonzero real number such that x^2 is rational, let $k \in \mathbf{Q}$, $k \neq 0, -1, -2, \ldots$ be fixed, and suppose that $f_k(x) = 0$ or $f_{k+1}(x)/f_k(x)$ is rational. Then $f_k(x)$ and $f_{k+1}(x)$ are both integer multiples of the same quantity: say $f_k(x) = ay$ and $f_{k+1}(x) = by$ for integers a and b. We allow a or b to be zero. But y cannot be zero, since it would then follow from (2) that $f_{k+n}(x) = 0$ for every $n = 1, 2, \ldots$, which would contradict (3).

Let q be a positive integer such that (bq/k), (kq/x^2), and (q/x^2) are all integers. Now let $G_0 = f_k(x)$ and

$$G_n = \frac{q^n}{k(k+1) \cdots (k+n-1)} f_{k+n}(x) \quad (n = 1, 2, \ldots)$$

for each $n = 1, 2, \ldots$. Then $G_0 = ay$, $G_1 = (bq/k)y$, and from (2) we can calculate that

$$G_{n+2} = \left(\frac{kq}{x^2} + \frac{q}{x^2} n \right) G_{n+1} - \left(\frac{q^2}{x^2} \right) G_n \tag{4}$$

for every $n = 0, 1, \ldots$. The coefficients in (4) are integers, so G_n is an integer multiple of y for every n. Since $f_{k+n}(x) \to 1$ by (3) and $q^n/[k(k+1) \cdots (k+n-1)] \to 0$, we have $G_n \to 0$. But $f_{k+n}(x) \to 1$ also implies that G_n is positive for all sufficiently large n. Positive integer multiples of y cannot converge to zero. The contradiction means that $f_k(x)$ and $f_{k+1}(x)$ cannot both be integer multiples of the same quantity. ■

Corollary 2. *π^2 is irrational.*

Proof: $f_{1/2}(\pi/4) = \cos(\pi/2) = 0$. ∎

Corollary 3. *If $x \neq 0$ is rational, then $\tan x$ is irrational.*

Proof: Since $(x/2)^2$ is nonzero and rational, $f_{3/2}(x/2)/f_{1/2}(x/2) = (\tan x)/x$ is irrational, and then so is $\tan x$. ∎

Although we eliminated (1) from the proof, for the sake of completeness we give a simple and self-contained proof of (1) using (2). We prove that (1) holds for every complex number x. The continued fraction

$$\cfrac{b_1}{1+\cfrac{b_2}{1+\ddots\cfrac{}{1+\cfrac{b_{n-1}}{1+b_n}}}}$$

will be denoted by $[b_1, \dots, b_n]$. Since occasionally we may have to divide by zero, we add to the set $\mathbf{C}$ of complex numbers an infinite element ∞, and adopt the following conventions: (i) $a/0 = \infty$ ($a \in \mathbf{C}$, $a \neq 0$); (ii) $a/\infty = 0$, ($a \in \mathbf{C}$); and (iii) $a + \infty = a - \infty = \infty$ ($a \in \mathbf{C}$). It is easy to see, using induction on n, that

$$\text{if } |b_i| \leq 1/4 \text{ for every } i = 1, \dots, n-1 \text{ and if } |b_n| \leq 1/2, \quad \text{then } |[b_1, \dots, b_n]| \leq 1/2. \tag{5}$$

We show next that if $|b_i| \leq 1/4$ for every $i = 1, \dots, n$ and if $|\delta| \leq 1/4$, then

$$|[b_1, \dots, b_{n-1}, b_n + \delta] - [b_1, \dots, b_n]| \leq |\delta|. \tag{6}$$

This is clearly true for $n = 1$. Let $n > 1$, and suppose (6) is true for $n - 1$. Let $|b_i| \leq 1/4$ $(i = 1, \dots, n)$ and $|\delta| \leq 1/4$. Denoting $u = [b_2, \dots, b_n]$ and $v = [b_2, \dots, b_{n-1}, b_n + \delta]$, we have $|u|, |v| \leq 1/2$ by (5), and $|v - u| \leq |\delta|$ by the induction hypothesis. Then

$$\begin{aligned} |[b_1, \dots, b_{n-1}, b_n + \delta] - [b_1, \dots, b_n]| &= \left| \frac{b_1}{1+v} - \frac{b_1}{1+u} \right| \\ &= \left| \frac{b_1(v-u)}{(1+u)(1+v)} \right| \\ &\leq \frac{(1/4)|\delta|}{(1-(1/2))(1-(1/2))} = |\delta|, \end{aligned}$$

which completes the proof.

Now let $x \neq 0$ be fixed. Let $k = 1/2$, and put $a_n = f_{n+(3/2)}(x)/f_{n+(1/2)}(x)$ $(n = 0, 1, \dots)$. By (2) we have

$$a_n = \cfrac{1}{1 - \cfrac{x^2}{(n+(1/2))(n+(3/2))} a_{n+1}} \qquad (n = 0, 1, \dots).$$

Since $a_0 = \tan(2x)/(2x)$, this implies

$$\tan(2x)/(2x) = \left[1, -\frac{x^2}{(1/2)\cdot(3/2)}, \frac{x^2}{(3/2)\cdot(5/2)}, \dots, -\frac{x^2}{(n-(1/2))(n+(1/2))}a_n\right]$$

for every n. Replacing x by $x/2$, and multiplying by x, we obtain

$$\tan x = \left[x, -\frac{x^2}{1\cdot 3}, \frac{x^2}{3\cdot 5}, \dots, -\frac{x^2}{(2n-1)(2n+1)}a_n\right]. \tag{7}$$

Let $N > 1$ be such that $x^2/((2n-1)(2n+1)) < 1/4$ and $a_n \in (0, 2)$ for every $n \geq N$ (recall that $\lim_{n\to\infty} a_n = 1$ by (3)). Let

$$P_n = \left[-\frac{x^2}{(2N+1)(2N+3)}, -\frac{x^2}{(2N+3)(2N+5)}, \dots, -\frac{x^2}{(2n-1)(2n+1)}\right]$$

and

$$Q_n = \left[-\frac{x^2}{(2N+1)(2N+3)}, -\frac{x^2}{(2N+3)(2N+5)}, -\frac{x^2}{(2n-1)(2n+1)}a_n\right]$$

for every $n > N$. Then (6) ensures that $|P_n - Q_n| \leq |a_n - 1|$ for every $n > N$. Let

$$F_N(y) = \left[x, -\frac{x^2}{1\cdot 3}, -\frac{x^2}{3\cdot 5}, \dots, -\frac{x^2}{(2N-1)(2N+1)}, y\right].$$

It is easy to check that F_N (as a function of y) is a homeomorphism of $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ onto itself; in fact, F_N is a fractional linear transformation. Since $\tan x = F_N(Q_n)$ by (7), we have $Q_n = F_N^{-1}(\tan x)$ for every $n > N$. Since $|P_n - Q_n| \leq |a_n - 1| \to 0$ as $n \to \infty$, this implies $\lim_{n\to\infty} P_n = F_N^{-1}(\tan x)$, and hence, by the continuity of F_N, we obtain $\tan x = \lim_{n\to\infty} F_N(P_n)$. However,

$$F_N(P_n) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{\ddots\, (2n-1) - \cfrac{x^2}{2n+1}}}} \stackrel{\text{def}}{=} R_n.$$

Since the right hand side of (1) is defined as $\lim_{n\to\infty} R_n$, this proves (1). ∎

Note that (1) is valid for every $x \in \mathbf{C}$, even for $x = (\pi/2) + k\pi$, when $\tan x$ is to be interpreted as ∞. The conventions concerning ∞ may be needed for other values of x, too, in order to compute some of the "convergents" R_n. (Take, for example, $x = \sqrt{3}$.) However, for every given x, R_n can be computed using finite numbers only, if n is large enough. Indeed, it is easy to see that there is a finite set S (depending on x) such that for every $y \notin S$, the computation of $F_N(y)$ does not involve ∞. Since $\lim_{n\to\infty} a_n = 1$, it follows from (2) that $a_n \neq 1$ if $n > n_0$. From this one can prove that $P_n \neq Q_n$ for $n > n_0$, and hence every number occurs in the sequence P_n only a finite number of times. This implies that for $n > n_1$ we have $P_n \notin S$, and then the computation of $R_n = F_N(P_n)$ needs finite numbers only.

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...he talked of a learned monk of many centuries ago, who did hit upon a way of multiplying numbers. That in itself I might understand, for it was simple, but the adding of each last two figures to make the next. To wit, one, two, three, five, eight, thirteen, one and twenty, and thus forward as you may will. Mr B. averred that he himself did believe these numbers appeared, though secretly, in many places in nature, as it were a divine cipher that all living things must copy, for that the ratio between its successive numbers was that also of a secret of the Greeks, who did discover a perfect proportion, I believe he said it to be of one to one and six tenths. He pointed to all that chanced about us, and said that these numbers might be read therein; and cited other examples, that I forget now except that many accorded with the order of petals and leaves in trees and herbs, I know not what.

John Fowles, *A Maggot*, New American Library, 1985
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