We could have required that (2) holds for all $f_i \prec g_i$ (this relation defined by (3) and (4)), dropping the assumption that f_i, g_i are decreasing. This gives, as in Theorem 1, (a), the relation (11) with the h_j now of arbitrary sign. It follows that $\Phi(u_i+h_i)-\Phi(u_i)$ and therefore also $\partial\Phi/\partial u_i$ does not depend on the u_j with $j\neq i$. Integrating we get $\Phi=\phi(t, u_i)+\psi(t, U_J)$. In this way we obtain

THEOREM 2. *A function Φ satisfies* (2) *with f_i, g_i not necessarily decreasing if and only if Φ is of the form*

$$\Phi(t, u_1, \cdots, u_n) = \sum_{i=1}^{n} \Phi_i(t, u_i),$$

where each Φ_i is a function of two variables which satisfies (6) *and* (7).

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A PROOF OF THE IRRATIONALITY OF π

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Assume $\pi=a/b$, a and b integers. Then, with $N=2a$, $\sin N=0$, $\cos N=1$, and $\cos(N/2)=\pm 1$.

If m is zero or a positive integer, then

$$A_m(x) \equiv \sum_{k=0}^{\infty} (-1)^k(2k+1)^m \frac{x^{2k+1}}{(2k+1)!} = P_m(x)\cos x + Q_m(x)\sin x$$

where $P_m(x)$ and $Q_m(x)$ are polynomials in x with integral coefficients. (Proof by induction on m: $A_{m+1}=xdA_m/dx$, and $A_0=\sin x$.)

Thus $A_m(N)$ is an integer for every positive integer m.

If t is any positive integer, then

$$\begin{aligned} B_t(N) &\equiv \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1-t-1)(2k+1-t-2)\cdots(2k+1-2t)}{(2k+1)!} N^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^t - b_1(2k+1)^{t-1} + \cdots \pm b_t}{(2k+1)!} N^{2k+1} \\ &= A_t(N) - b_1A_{t-1}(N) + \cdots \pm b_tA_0(N). \end{aligned}$$

Since all the b_i are integers, $B_t(N)$ must be an integer too.

Now

$$B_t(N) = \sum_{k=0}^{[(t-1)/2]} + \sum_{k=[(t+1)/2]}^{t-1} + \sum_{k=t}^{\infty}.$$

In the first sum, the numerator of each fraction is a product of t consecutive integers, therefore it is divisible by $t!$, and therefore by $(2k+1)!$ since $2k+1 \leqq t$. Thus each term of the first sum is an integer. Each term of the second sum is zero. Thus the third sum must be an integer, for every positive integer t.

This third sum is

$$\sum_{k=t}^{\infty} (-1)^k \frac{(2k-t)!}{(2k+1)!(2k-2t)!} N^{2k+1}$$
$$= (-1)^t \frac{t!}{(2t+1)!} N^{2t+1}\left(1 - \frac{(t+1)(t+2)}{(2t+2)(2t+3)} \frac{N^2}{2!} + \frac{(t+1)(t+2)(t+3)(t+4)}{(2t+2)(2t+3)(2t+4)(2t+5)} \frac{N^4}{4!} - \cdots\right).$$

Let $S(t)$ stand for the sum in the parenthesis. Certainly

$$|S(t)| < 1 + N + \frac{N^2}{2!} + \cdots = e^N.$$

Thus the whole expression is absolutely less than

$$\frac{t!}{(2t+1)!} N^{2t+1} e^N < \frac{N^{2t+1}}{t^{t+1}} e^N < (N^2/t)^{t+1} e^N;$$

for $t > t_0$, this is certainly less than 1.

Therefore necessarily $S(t) = 0$ for every integer $t > t_0$. But this is impossible, because

$$\lim_{t\to\infty} S(t) = 1 - \frac{1}{2^2}\cdot\frac{N^2}{2!} + \frac{1}{2^4}\cdot\frac{N^4}{4!} - \cdots = \cos(N/2) = \pm 1.$$

It can be proved similarly that the natural logarithm of a rational number must be irrational: From $\log(a/b) = c/d$ would follow $e^c = a^d/b^d = A/B$. Then

$$B \cdot \sum_{k=0}^{\infty} \frac{(k-t-1)(k-t-2)\cdots(k-2t)}{k!} c^k$$

would have to be an integer for every positive integer t, and a contradiction results, as before.