

In this note we will discuss the limit inferior and limit superior of a sequence in a bit more detail than in the text. Both the limit inferior and the limit superior may be viewed as a replacement for the limit in many situations. They have the advantage that they always exist. Thus we may separate the question of existence of a potential limit and its estimation. However, their usefulness is much greater than that.

The limit inferior and the limit superior do not replace the notion of limit however, because they fail to have the crucial additive and multiplicative properties of limits.

We will make use of the affine notions of infinity, $+\infty$ and $-\infty$, throughout this note (as opposed to the projective notion of infinity ∞ popular in dealing with complex numbers). We make the usual conventions regarding order (if $\alpha \in \mathbb{R}$ then $\alpha < +\infty$) and arithmetic (if $\alpha \in \mathbb{R}$ then $\alpha + \infty = +\infty$). Some care is needed of course since some expressions are not defined (for example $+\infty - \infty$). Following tradition I may be careless and write ∞ for $+\infty$ and I may neglect to mention special cases that need to be checked.

The word *number* will mean *real number* throughout (unless qualified). In particular $+\infty$ and $-\infty$ are not *numbers*. We will use the term *extended real number* to indicate a real number, $+\infty$ or $-\infty$.

1. SUP AND INF

The completeness property of the real numbers may be formulated as *each nonempty set A of real numbers which has an upper bound, has a least upper bound, $\text{lub } A$* . An equivalent statement is *each nonempty set A of real numbers which has a lower bound, has a greatest lower bound, $\text{glb } A$* .

It is convenient to extend the $\text{lub } A$ and $\text{glb } A$ notions. Hence for any set of real numbers A we define the *supremum* of A by

$$(1) \quad \sup A = \begin{cases} -\infty & \text{if } A = \emptyset \\ +\infty & \text{if } A \text{ has no upper bound} \\ \text{lub } A & \text{elsewhere.} \end{cases}$$

We define the *infimum* of A in like manner:

$$(2) \quad \inf A = \begin{cases} +\infty & \text{if } A = \emptyset \\ -\infty & \text{if } A \text{ has no lower bound} \\ \text{glb } A & \text{elsewhere.} \end{cases}$$

The peculiar definitions for the empty set keep things consistent, but sometimes are a nuisance. Note for example $\inf A \leq \sup A$ if and only if $A \neq \emptyset$.

2. LIMITS OF SEQUENCES

Let $(a_n)_{n \geq 1}$ be a sequence of real numbers. Recall the sequence *converges* to the real number L if

$$(3) \quad \forall \epsilon > 0 \exists N \text{ such that } n > N \text{ implies } |a_n - L| < \epsilon.$$

We also say that $(a_n)_{n \geq 1}$ *has limit* L and we write $\lim_{n \rightarrow \infty} a_n = L$.

A sequence can diverge in many way. We single out two: divergence to $+\infty$ and divergence to $-\infty$. We can include these two cases in our notion of limit by rephrasing the definition above.

Let $(a_n)_{n \geq 1}$ be a sequence of real numbers. We say the sequence *has limit* the extended real number L if

$$(4) \quad \forall s \in \mathbb{R} \text{ such that } s < L, \forall t \in \mathbb{R} \text{ such that } L < t, \{n \mid a_n < s \text{ or } t < a_n\} \text{ is a finite set.}$$

Note if L is infinite, then the condition on s or the condition on t is not satisfied by any real number, and the corresponding condition on a_n is taken to be vacuous. Write out the two possible cases and make sure you understand them.

Recall that another way to formulate the completeness of the real numbers is by the assertion that *each bounded monotone sequence converges*. With our extended notion of limit we can now assert each monotone sequence has a limit. The limit is finite if and only if the monotone sequence is bounded (in which case it converges to the limit).

Note for a monotone increasing sequence $(a_n)_{n \geq 1}$

$$(5) \quad \lim_{n \rightarrow \infty} a_n = \sup \{a_n \mid n \geq 1\} = \sup_{n \geq 1} a_n$$

and for a monotone decreasing sequence $(b_n)_{n \geq 1}$

$$(6) \quad \lim_{n \rightarrow \infty} b_n = \inf \{a_n \mid n \geq 1\} = \inf_{n \geq 1} b_n$$

Here is an important variation on the definition (4) – we say that the extended real number W is an *accumulation point* of the sequence $(a_n)_{n \geq 1}$ if

$$(7) \quad \forall s \in \mathbb{R} \text{ so } s < W, \forall t \in \mathbb{R} \text{ so } W < t, \{n \mid s < a_n \text{ and } a_n < t\} \text{ is an infinite set.}$$

Note as before if either s or t does not exist then the corresponding condition on a_n is vacuous. Thus $+\infty$ is an accumulation point if and only if for each real number s we have $\{n \mid s < a_n\}$ is an infinite set.

Exercise 1. Show that W is an accumulation point of the sequence $(a_n)_{n \geq 1}$ if and only if there is a subsequence $(a_{n_k})_{k \geq 1}$ with limit W .

3. LIMIT SUPERIOR AND LIMIT INFERIOR

Let $(a_n)_{n \geq 1}$ be a sequence of real numbers. Define

$$(8) \quad A_n = \inf_{k \geq n} a_k$$

$$(9) \quad B_n = \sup_{k \geq n} a_k.$$

Then $-\infty \leq A_n < +\infty$ and the A_n form an increasing sequence of real numbers or $A_n = -\infty$ for each n . Also, $-\infty < B_n \leq +\infty$ and the B_n form a decreasing sequence of real numbers or $B_n = +\infty$ for each n . Thus either $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ are monotone sequences of real numbers, and so have a limit, or are constant sequences of $-\infty$ or $+\infty$, to which we assign the obvious limit. Thus the following definitions make sense:

$$(10) \quad \liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \inf_{k \geq n} a_k = \sup_{n \geq 1} \inf_{k \geq n} a_k$$

$$(11) \quad \limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} \sup_{k \geq n} a_k = \inf_{n \geq 1} \sup_{k \geq n} a_k.$$

Clearly

$$-\infty \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq +\infty$$

and these quantities always exist.

Proposition 1. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let A be an extended real number. Then

$$\liminf_{n \rightarrow \infty} a_n \leq A$$

if and only if for each real number t with $A < t$ the set $\{n \mid a_n < t\}$ is infinite.

Proof. If $A = +\infty$ then there is no t with $A < t$ and so for each such t anything is true. We may now assume $-\infty \leq A < +\infty$.

Suppose $\liminf_{n \rightarrow \infty} a_n \leq A$. Let $t \in \mathbb{R}$ and suppose $A < t$. Since $\sup_{n \geq 1} \inf_{k \geq n} a_k \leq A < t$ we have

$$\inf_{k \geq n} a_k \leq A < t$$

for each $n \geq 1$. It follows for each $n \geq 1$ there exists $k_n \geq n$ such that

$$a_{k_n} < t,$$

(since otherwise t would be a lower bound for $\{a_n, a_{n+1}, \dots\}$ and so $t \leq \inf_{k \geq n} a_k$). Since $k_n \geq n$ for each n we see that the set $\{n \mid a_n < t\}$ is infinite.

Conversely suppose $A < t$ implies $\{n \mid a_n < t\}$ is infinite. Let $A < t$. Then for each $n \geq 1$ there is $k_n \geq n$ such that $a_{k_n} < t$. Thus for each $n \geq 1$ we have $\inf_{k \geq n} a_k \leq t$. The least upper bound of this monotone increasing sequence is therefore bounded by t . Thus $\limsup_{n \rightarrow \infty} a_n \leq t$. But this inequality has been shown to hold for each t with $A < t$. It must follow that $\limsup_{n \rightarrow \infty} a_n \leq A$ (since for any number s if $A < s$ then $t < s$ for some t with $A < t$). \square

Proposition 2. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let A be an extended real number. Then

$$A \leq \liminf_{n \rightarrow \infty} a_n$$

if and only if for each real number s with $s < A$ the set $\{n \mid a_n < s\}$ is finite.

Proof. If $A = -\infty$ then there is no s with $s < A$ and so for each such s anything is true. We may now assume $-\infty < A \leq +\infty$.

Suppose $A \leq \liminf_{n \rightarrow \infty} a_n$. Let $s \in \mathbb{R}$ and suppose $s < A$. Then $s < \sup_{n \geq 1} \inf_{k \geq n} a_k$ implies s is not an upper bound for the numbers $\inf_{k \geq n} a_k$ (since the sup is the least upper bound). It follows there exists N such that $s < \inf_{k \geq N} a_k$. But then $\{n \mid a_n < s\} \subseteq \{1, 2, \dots, N\}$.

Conversely suppose $s \in \mathbb{R}$ and $s < A$ imply $\{n \mid a_n < s\}$ is finite. Let $s < A$. Then there exists n such that $k \geq n$ implies $a_k \geq s$, that is, s is a lower bound for $\{a_n, a_{n+1}, \dots\}$. It follows that $\inf_{k \geq n} a_k \geq s$. This certainly implies that $\liminf_{n \rightarrow \infty} a_n \geq s$. Since we proved this assertion for each s with $s < A$ we must have $\liminf_{n \rightarrow \infty} a_n \geq A$. \square

Theorem 3. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let A be an extended real number. Then

$$A = \liminf_{n \rightarrow \infty} a_n$$

if and only if whenever $s \in \mathbb{R}$, $s < A$ and $t \in \mathbb{R}$, $A < t$ then

$$\{n \mid a_n < s\} \text{ is finite and } \{n \mid a_n < t\} \text{ is infinite.}$$

In particular $\liminf_{n \rightarrow \infty} a_n$ is an accumulation point of $(a_n)_{n \geq 1}$.

Proof. The first part follows by proposition 1 and 2. The last part follows by exercise 1. \square

Proposition 4. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let B be an extended real number. Then

$$B \leq \limsup_{n \rightarrow \infty} a_n$$

if and only if for each real number s with $s < B$ the set $\{n \mid a_n > s\}$ is infinite.

Proof. Indeed $\limsup_{n \rightarrow \infty} a_n = -\liminf_{n \rightarrow \infty} -a_n$. Now use proposition 1 \square

Proposition 5. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let B be an extended real number. Then

$$\limsup_{n \rightarrow \infty} a_n \leq B$$

if and only if for each real number t with $B < t$ the set $\{n \mid a_n < t\}$ is finite.

Proof. Indeed $\limsup_{n \rightarrow \infty} a_n = -\liminf_{n \rightarrow \infty} -a_n$. Now use proposition 2. \square

Theorem 6. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let B be an extended real number. Then

$$B = \limsup_{n \rightarrow \infty} a_n$$

if and only if whenever $s \in \mathbb{R}$, $s < B$ and $t \in \mathbb{R}$, $B < t$ then

$$\{n \mid a_n > s\} \text{ is infinite and } \{n \mid a_n > t\} \text{ is finite.}$$

In particular $\limsup_{n \rightarrow \infty} a_n$ is an accumulation point of $(a_n)_{n \geq 1}$.

Proof. Indeed $\limsup_{n \rightarrow \infty} a_n = -\liminf_{n \rightarrow \infty} -a_n$. Now use theorem 3. \square

Corollary 7. The sequence $(a_n)_{n \geq 1}$ of real numbers has a limit if and only if $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$. Moreover, in this case

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

If we let $L = \liminf_{n \rightarrow \infty} a_n$ and $U = \limsup_{n \rightarrow \infty} a_n$ then the following table summarizes the properties of L and U :

finite	-		L		---		U		-	finite
infinite	-	-	-	-	-		---		-	infinite

Exercise 2. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers and let S be the set of accumulation points of $(a_n)_{n \geq 1}$. Prove

$$\limsup_{n \rightarrow \infty} a_n = \sup S \quad \text{and} \quad \liminf_{n \rightarrow \infty} a_n = \inf S.$$

Exercise 3. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be sequences of real numbers. Prove

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

provided right side is not $+\infty - \infty$.

Exercise 4. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be sequences of real numbers. Prove

$$\liminf_{n \rightarrow \infty} (a_n + b_n) \geq \liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n.$$

provided right side is not $+\infty - \infty$.

Exercise 5. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be bounded sequences of real numbers. Suppose $a_n \geq 0$ and $b_n \geq 0$ for each $n \geq 1$. Prove

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \left(\limsup_{n \rightarrow \infty} a_n \right) \left(\limsup_{n \rightarrow \infty} b_n \right).$$

Exercise 6. Show the result of the previous exercise is true for unbounded sequences provided we assume the individual limit superiors are strictly positive.

Here is a useful result related to the results of the exercises above.

Proposition 8. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be sequences of real numbers. Suppose $a_n > 0$ and $b_n \geq 0$ for each $n \geq 1$. Suppose moreover that $a = \lim_{n \rightarrow \infty} a_n$ exists and suppose $0 < a < \infty$. Then

$$\limsup_{n \rightarrow \infty} (a_n b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\limsup_{n \rightarrow \infty} b_n \right).$$

Proof. From the exercises we have

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \left(\lim_{n \rightarrow \infty} a_n \right) \left(\limsup_{n \rightarrow \infty} b_n \right).$$

Now since $b_n = \frac{1}{a_n} (a_n b_n)$ and $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}$ we also have

$$\limsup_{n \rightarrow \infty} b_n \leq \frac{1}{a} \limsup_{n \rightarrow \infty} (a_n b_n).$$

□

As an important special case we have if $c > 0$ and $b_n \geq 0$ then

$$(12) \quad \limsup_{n \rightarrow \infty} c^{1/n} b_n = \limsup_{n \rightarrow \infty} b_n.$$

Here is an interesting result which will be important to us later in our study of convergence of series.

Theorem 9. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers with $a_n > 0$ for each $n \geq 1$. Then

$$(13) \quad \liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} a_n^{1/n} \leq \limsup_{n \rightarrow \infty} a_n^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

Proof. It suffices to prove

$$(14) \quad \limsup_{n \rightarrow \infty} a_n^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

Let

$$L = \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

If $L = +\infty$ there is nothing to prove. Thus we may assume $0 \leq L < +\infty$. Let $t > L$. Then the set $\{n \mid a_{n+1}/a_n > t\}$ is finite. It follows there exists N such that $a_{n+1} \leq ta_n$ for $n \geq N$. By induction

$$a_{N+k} \leq t^k a_N \quad \text{for } k \geq 1.$$

We can rewrite this as

$$a_n \leq t^n (t^{-N} a_N) \quad \text{for } n \geq N.$$

Now let c be the maximum of the finite set

$$\{t^{-1}a_1, t^{-2}a_2, \dots, t^{-N}a_N\}.$$

Then by the estimate above we have

$$a_n \leq t^n c \quad \text{for } n \geq 1.$$

It follows

$$\limsup_{n \rightarrow \infty} a_n^{1/n} \leq \limsup_{n \rightarrow \infty} tc^{1/n} = t.$$

Since we proved this inequality for each $t > L$ it follows that $\limsup_{n \rightarrow \infty} a_n^{1/n} \leq L$. □

Exercise 7. Let

$$a_n = 2^{(-1)^n - n}.$$

Prove

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}, \quad \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2, \quad \lim_{n \rightarrow \infty} a_n^{1/n} = \frac{1}{2}.$$

Exercise 8. Without making use of Stirling's prove

$$\lim_{n \rightarrow \infty} \left(\frac{(2n)!}{(n!)^2} \right)^{\frac{1}{n}} = 4.$$

Exercise 9. Let $(a_n)_{n \geq 1}$ be a sequence of real numbers with $a_n > 0$ for each $n \geq 1$. Consider the Cesaro means of the sequence

$$\sigma_n = \frac{1}{n} \sum_{k=1}^n a_k.$$

Prove

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} \sigma_n \leq \limsup_{n \rightarrow \infty} a_n.$$

Conclude if $\lim_{n \rightarrow \infty} a_n$ exists then the Cesaro limit $\lim_{n \rightarrow \infty} \sigma_n$ exists and

$$\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} a_n.$$

Exercise 10. Compute the Cesaro limit of the sequence $(a_n)_{n \geq 1}$ defined by

$$a_n = 2 + (-1)^n.$$

Remark. These notes were thrown together very quickly and may contain some errors. I would be very pleased to receive corrections (and suggestions).

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