Chapter 1

Introduction

1.1 Motivation for Elimination Theory

To give a flavor of elimination theory and motivate its study, we start by discussing the following problem which involves geometric reasoning. Heymann in [25] posed the question of whether a triangle can be drawn using only a compass and a ruler, given the lengths of any three of its angle bisectors. One specific case of this problem can be mathematically formulated as follows.

Let ABC (see Figure 1.1) be the triangle in question, in which a, b and c are the lengths of the three sides, a_i the internal bisector of angle A, and a_e and b_e be the external bisectors of angles A and B, respectively. Heymann's question translates to whether ABC can be drawn, given a_i, a_e and b_e.

Equations (1.1) express the bisectors in terms of the sides and can be easily derived using high-school mathematics. To construct the triangle, one must be able to construct line segments of length a, b and c. From Corollary[1] 2 of Theorem 5.4.1 in [24], it follows that if the relationship between the three bisectors and a is an irre-

[1]The corollary says that if a real number α satisfies an irreducible polynomial over the field of rational of degree k, and if k is not a power of two, then a line segment of length α cannot be constructed using a compass and a ruler.

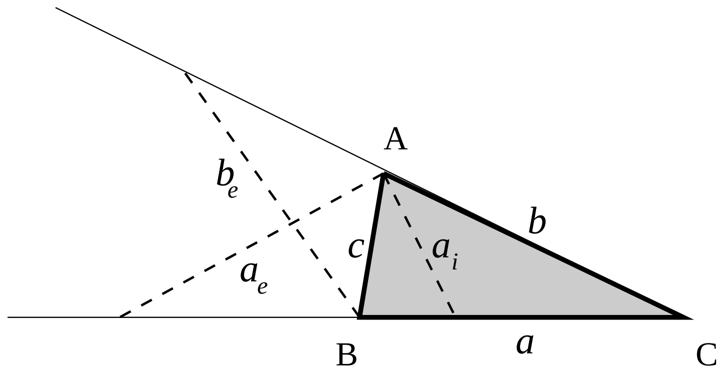

Figure 1.1: Triangle for Heymann's Problem

$$\left.\begin{aligned} a_i^2 &= \frac{cb(c+b-a)(c+b+a)}{(b+c)^2} \\ a_e^2 &= \frac{cb(a+b-c)(c-b+a)}{(c-b)^2} \\ b_e^2 &= \frac{ac(a+b-c)(c+b-a)}{(c-a)^2} \end{aligned}\right\} (1.1)$$

$$\left.\begin{aligned} a_i^2(b+c)^2 - cb(c+b-a)(c+b+a) &= 0 \\ a_e^2(c-b)^2 - cb(a+b-c)(c-b+a) &= 0 \\ b_e^2(c-a)^2 - ac(a+b-c)(c+b-a) &= 0 \end{aligned}\right\} (1.2)$$

Figure 1.2: Equations for Heymann's Problem

$$
\left[\begin{matrix}
0 & 0 & 0 & 3600 + 136a^2 & -128 & -136 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-136a^2 - 3600 & 0 & 256a^2 & 1664a^2 - 3600 + 68a^4 & -64a^2 + 128 & 136 - 52a^2 \\
0 & -136a^2 - 3600 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -7200 + 628a^2 + 34a^4 & -32a^2 + 256 & -34a^2 + 272 \\
0 & 128a^2 & -256a + 32a^3 & -104a^3 + 1088a - 4a^5 & 0 & 4a^3 + 36a \\
0 & 136a^2 & 628a + 68a^3 & -32a^3 + 256a & -8a^3 - 36a & -32a \\
0 & 136a^2 & -1172a & -256a + 32a^3 & 68a & 0 \\
136 & 0 & 0 & 136 - 68a^2 & 0 & 0 \\
0 & 0 & 3600 - 1664a^2 - 68a^4 & -256a^2 & 240a^2 - 544 + 8a^4 & -128 + 64a^2 \\
136 & 0 & 0 & 136 - 84a^2 & 0 & 0 \\
128 & 0 & 16a^2 & -64a^2 + 128 & -8a^2 + 16 & 0 \\
0 & 0 & 0 & 4a^3 - 32a & 0 & -4a
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & -136 & 0 & 0 \\
0 & 0 & 0 & -34a^2 - 900 & 0 & 34 & 32 \\
-628a^2 + 7200 - 34a^4 & 34a^3 - 272a & -256a + 32a^3 & -1172a & 136 - 68a^2 & 34a & 32a \\
0 & 128 & 16a^2 + 544 & -128 & 0 & 0 & -16 \\
3600 + 136a^2 & -136a & -128a & -136a & -34a^2 + 272 & 0 & 0 \\
-544a - 16a^3 & 136 - 52a^2 & 0 & 188a^2 - 408 + 8a^4 & -68a - 4a^3 & 0 & 0 \\
-128a & 0 & 136 - 84a^2 & -128 + 64a^2 & 0 & 0 & 8a^2 - 16 \\
128a & 0 & 136 - 68a^2 & -64a^2 + 128 & 32a & 0 & 0 \\
34a^2 - 272 & -34a & 0 & 34a & 0 & 0 & 0 \\
0 & 0 & 1800a + 68a^3 & 512a - 64a^3 & -128 + 64a^2 & -32a & -104a - 8a^3 \\
34a^2 - 272 & 34a & -64a & 134a - 4a^3 & 8a^2 - 16 & -4a & 0 \\
-256 + 32a^2 & 0 & -168a & 64a & 0 & 0 & 8a \\
16a & 0 & 0 & -8a^2 + 16 & 4a & 0 & 0
\end{matrix}\right]
$$

Figure 1.3: Resultant Matrix for Heymann's Problem

ducible polynomial whose degree in a is *not* 2^m for some integral m, then the answer to Heymann's question is in the negative. In other words, we can solve Heymann's problem if we can:

Derive the relationship purely between a, a_i, a_e and b_e.

A straight-forward way of accomplishing this derivation is by eliminating b and c from Equations (1.2) (which follow from Equations (1.1)). Computationally, this is not an easy task and may require quite a bit of computational resources as well as innovative techniques. It is such techniques that are the object of study under the general subject of *Elimination Theory.* Elimination theory studies and develops efficient techniques for symbolically solving systems of non-linear polynomial equations.

To give a flavor of the contributions of this thesis, we give a solution to Heymann's problem using a technique developed in this thesis. It can be shown using the results of this thesis that the polynomial in a, a_i, a_e and b_e which is the result of eliminating b and c from the three equations (1.2) is in fact the determinant of a 13×13 matrix (whose entries are polynomials in a, a_i, a_e and b_e). Since this matrix is very large and we are anyway interested only in knowing the degree of its determinant in a, we give the matrix after substituting $a_i = 3, a_e = 5$ and $b_e = 2$ in Figure 1.3. Its determinant has degree 20 in a. Since 20 is not an integral power of 2, the answer to Heymann's problem is in the negative. Thus, in general, it is impossible to draw a triangle using just a compass and a ruler, if the lengths of any three of its angle bisectors are given.

1.1.1 Elimination Theory

Elimination theory is the study of techniques and algorithms for eliminating variables from a set of polynomial equations. In the previous section we demonstrated how one problem involving geometric reasoning can be solved using elimination theory. In fact, elimination of variables is a fundamental problem which often comes up in many areas of mathematics, engineering, physical and computer sciences. Many practical problems in robotics, computer vision, computational biology, solid modeling, mechanical and chemical engineering, thermodynamics, physics etc. can be reformulated as a problem of eliminating a subset of variables from a set of polynomial equations. For example, Hoffman in [26] posed the challenging problem of deriving the implicit equation (in x, y and z coordinates) of a bicubic surface (which is plotted in Figure 1.4) from its parameterization. The problem of deriving implicit equations from parameterizations is known as the *implicitization problem*, and can be reformulated as an elimination problem. The bicubic implicitization problem posed by Hoffman translates to the problem of eliminating variables s and t from the fol-

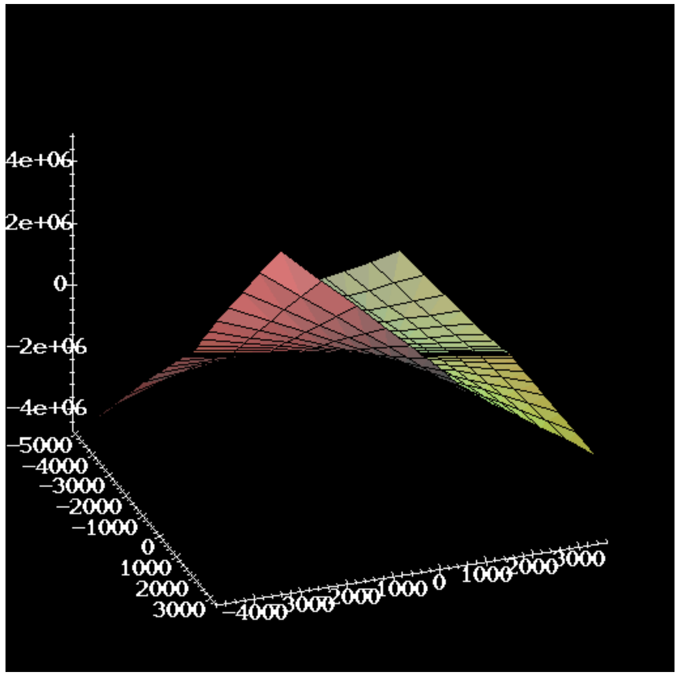

Figure 1.4: A Bicubic Surface

lowing three polynomials:

$$
\begin{aligned}
q_1 &= 3\,t\,(t-1)^2 + (s-1)^3 + 3\,s - x \\
q_2 &= 3\,s\,(s-1)^2 + t^3 + 3\,t - y \\
q_3 &= -3\,s\left(s^2 - 5\,s + 5\right)t^3 - 3\left(s^3 + 6\,s^2 - 9\,s + 1\right)t^2 \\
&\quad + t\left(6\,s^3 + 9\,s^2 - 18\,s + 3\right) - 3\,s\,(s-1) - z.
\end{aligned}
$$

The problem of eliminating variables from polynomial equations is inherently hard in terms of computational resources. Nevertheless, as computer technology and symbolic manipulation techniques have drastically improved over the last few decades, elimination problems in many real applications have come within the grasp of state-of-the-art computational power. For instance, the bicubic implicitization problem stated above had proven to be a computationally challenging problem when Hoffman [26] posed it in his 1990 paper. He reported that even the state of the art elimination methods in 1990 required time of the order of 10^5 seconds to implicitize the surface, which is quite unacceptable for real life applications. However, today, with the development of fast variable elimination techniques such as those in this thesis, this problem can easily be solved within a few ($< 10^1$) seconds on desktop workstations (the bicubic surface represented by the above equations is plotted in Figure 1.4). Already, the fastest solution to problems such as real-time inverse kinematics for a general *6R* serial robot arm manipulator [33] use specialized techniques from elimination theory. Such successes have inspired a renewed interest in developing efficient methods for eliminating variables and solving nonlinear equations. Evidence to this is provided by recent breakthroughs in elimination theory such as the development of fast algorithms for solving polynomial systems with finitely many solutions [18], resurrection and variations of classical constructive techniques for eliminating variables [5, 33], development of elimination methods which exploit the structure of polynomial systems to solve them efficiently [14, 7, 40], and the development of efficient techniques for numerically solving non-linear systems [35, 42].

Three major techniques for eliminating variables symbolically are:

1. Gröbner basis computations proposed by Buchberger [4],

2. Characteristic set computations proposed by Ritt [37] and

3. Resultant computations based on methods developed early this century by researchers such as Macaulay [32], Cayley [8], Bezout, Dixon [12], Hurwitz etc.

A Gröbner basis of a polynomial ideal is a basis with many useful properties and provides answers to most ideal-theoretic questions about the ideal. The first algorithm to compute Gröbner basis of ideals was given by Buchberger in [4] and since then, many efficient variations have appeared [2]. Among other things, Gröbner bases can be used to find solutions to a set of polynomials, compute projections of their variety, eliminate variables and test polynomials for ideal membership. A variation on the Gröbner basis algorithm for 0-dimensional ideals is what was used by Hoffman to solve the bicubic implicitization problem in 10^5 seconds in 1990 [26].

A characteristic set [10, 44] of a set of polynomials is a *triangular* set of polynomials with *almost* the same set of common solutions as the original. The first algorithm to compute characteristic sets was given by Ritt, and recently Wu [44] resurrected it. Characteristic sets are typically computed by eliminating variables sequentially in some predetermined order using successive pseudo-division of polynomials. Characteristic sets can be used, among other things, to compute projections of varieties, eliminate variables and derive conditions under which a polynomial equation follows from another set of polynomial equations. Their effectiveness in proving geometry theorems is well established. In his book ([10]), Chou details many theorem proving procedures using characteristic sets, and lists hundreds of theorems he was able to automatically prove using his implementation of these procedures. Gao and Wang [20] used specialized characteristic set algorithms to solve Heymann's problem in 19 hours on a SUN 4 workstation.

Resultant is a characterization of the projection of the variety of a given set of polynomials into a smaller set of variables. Methods which compute the resultant can thus be used to eliminate a subset of variables from a set of polynomials.

The interesting property of multivariate resultant methods is that they eliminate n variables *together*, instead of sequentially, from $n+1$ polynomials. The fundamental principle by which multivariate resultant methods work is by reducing a non-linear elimination problem to a linear one. That is why the derivation in Heymann's problem was reduced to computing the determinant in Figure 1.3. Such linearization of nonlinear problems enables the applicability of a vast array of linear algebra and linear equation solving techniques that have been developed in the last two centuries, to non-linear variable elimination. The effectiveness of multivariate resultant methods has been demonstrated in a variety of applications [9, 14, 28, 29, 33, 34, 39], and for some applications only the methods involving resultants offer an acceptable real-time solution [33]. For example, the implicit equation of the bicubic surface can be derived within 50 seconds (compared to 10^5 seconds using Gröbner bases), and Heymann's question can be resolved within 300 seconds using resultant based methods on a SPARC station 10 (compared to 19 hours using Characteristic set type methods).

This thesis proposes, analyzes, evaluates and applies a new method for computing resultants based on a classical formulation by Dixon [12]. Before discussing the contributions of this thesis, we outline the different algorithms available for computing the resultant and some desirable qualities in resultant methods. Here we give an informal perspective on multivariate resultant methods so as to create a setting in which the contributions of this thesis and their importance can be understood. We will formally review the various multivariate resultant methods in detail later in this thesis.

1.1.2 Resultants: An Informal Perspective

There are three major formulations which can be used to compute multivariate resultants – (i) The Macaulay formulation, (ii) The Sparse resultant formulation and (iii)