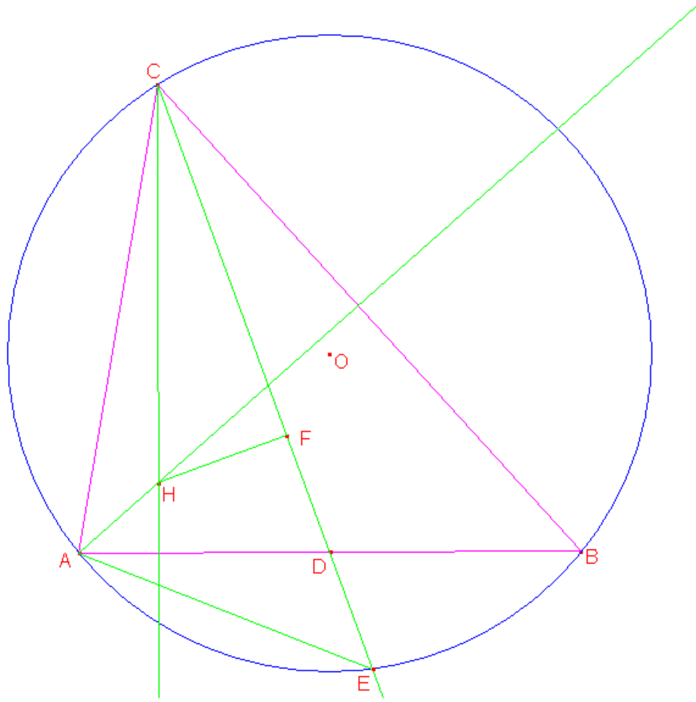
1. Dat je oštrougli trougao ABC. Neka je D središte stranice AB, i neka prava CD drugi put seče opisanu kružnicu trougla ABC u tački E. Tačka F je takva da je D središte duži EF. Dokazati da je $HF \perp CE$, gde je H ortocentar trougla ABC.



Rešenje (trigonometrija):

Neka je $\measuredangle BAC = \alpha$, $\measuredangle ABC = \beta$, a poluprečnik opisane kružnice R. Iz sinusne teoreme imamo da važe sledeće relacije:

$$a = 2R\sin\alpha,\ \ b = 2R\sin\beta,\ \ c = 2R\sin\gamma = 2R\sin(\alpha+\beta)$$

gde su a, b i c stranice trougla.

CD je težišna linija trougla ABC, pa je njena dužina jednaka:

$$CD = \sqrt{\frac{2a^2+2b^2-c^2}{4}} = \sqrt{\frac{8R^2\sin^2\alpha + 8R^2\sin^2\beta - 4R^2\sin^2(\alpha+\beta)}{4}}$$

$$CD = R\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)} \tag{1}$$

Primenjujući sinusnu teoremu na trougao BCD dalje imamo

$$\frac{\sin\measuredangle BCD}{\sin\beta} = \frac{BD}{CD} \iff \sin\measuredangle BCD = \frac{\frac{c}{2}\sin\beta}{CD}$$

pa smenom (1) u ovo dobijamo

$$\sin\measuredangle BCD = \frac{\sin\beta\sin(\alpha+\beta)}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} \tag{2}$$

Primetimo da je $\measuredangle DAE = \measuredangle BCD$ i $\measuredangle AED = \beta$ pa primenom sinusne teoreme na trougao ADE dobijamo sledeću relaciju:

$$\frac{DE}{AD} = \frac{\sin\measuredangle DAE}{\sin\measuredangle AED} \iff DE = \frac{\frac{c}{2}\sin\measuredangle BCD}{\sin\beta}$$

i uvrštajući (2) imamo

$$DE = \frac{R\sin^2(\alpha+\beta)}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} \tag{3}$$

Sada se iz (1) i (3) lako dobija

$$CF = CD - DE = R\frac{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta) - \sin^2(\alpha+\beta)}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}}$$

$$CF = 2R\frac{\sin^2\alpha + \sin^2\beta - \sin^2(\alpha+\beta)}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} \tag{4}$$

Pošto je $\measuredangle AHC = 180^o - \beta$ i $\measuredangle CAH = 90^o - \gamma = (\alpha+\beta) - 90^o$ iz sinusne teoreme primenjene na trougao ACH imamo

$$\frac{CH}{AC} = \frac{\sin\measuredangle CAH}{\sin\measuredangle AHC} \iff CH = -\frac{AC\cos(\alpha+\beta)}{\sin\beta}$$

odnosno

$$CH = -2R\cos(\alpha+\beta) \tag{5}$$

Da bi važilo $HF \perp CE$ potrebno je i dovoljno pokazati da je $\frac{CF}{CH} = \cos\measuredangle FCH$. Iz (4) i (5) imamo da je

$$\frac{CF}{CH} = \frac{2R\frac{\sin^2\alpha + \sin^2\beta - \sin^2(\alpha+\beta)}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}}}{-2R\cos(\alpha+\beta)} =$$

$$= -\frac{\sin^2\alpha + \sin^2\beta - \sin^2\alpha\cos^2\beta - \sin^2\beta\cos^2\alpha - 2\sin\alpha\sin\beta\cos\alpha\cos\beta}{\cos(\alpha+\beta)\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} =$$

$$= -\frac{\sin^2\alpha(1-\cos^2\beta) + \sin^2\beta(1-\cos^2\alpha) - 2\sin\alpha\sin\beta\cos\alpha\cos\beta}{\cos(\alpha+\beta)\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} =$$

$$= -\frac{2\sin\alpha\sin\beta(\sin\alpha\sin\beta - \cos\alpha\cos\beta)}{\cos(\alpha+\beta)\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} = \frac{2\sin\alpha\sin\beta\cos(\alpha+\beta)}{\cos(\alpha+\beta)\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}}$$

iz čega sledi

$$\frac{CF}{CH} = \frac{2\sin\alpha\sin\beta}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}} \tag{6}$$

Slično kao (2) može se pokazati da je

$$\sin\measuredangle ACD = \frac{\sin\alpha\sin(\alpha+\beta)}{\sqrt{2\sin^2\alpha + 2\sin^2\beta - \sin^2(\alpha+\beta)}}$$

Dalje dobijamo

$$\cos\measuredangle ACD=\sqrt{1-\sin^2\measuredangle ACD}=\sqrt{\frac{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)-\sin^2\alpha\sin^2(\alpha+\beta)}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{2\sin^2\alpha+2\sin^2\beta-\sin^2\alpha\cos^2\beta-\sin^2\beta\cos^2\alpha-2\sin\alpha\sin\beta\cos\alpha\cos\beta-\sin^2\alpha\sin^2(\alpha+\beta)}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{\sin^2\alpha+\sin^2\beta+2\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta-\sin^2\alpha\sin^2(\alpha+\beta)}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{\sin^2\alpha+\sin^2\beta+2\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta-\sin^4\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta\cos^2\alpha-2\sin^3\alpha\sin\beta\cos\alpha\cos\beta}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{\sin^2\alpha+\sin^2\beta+\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta-\sin^4\alpha+2\sin^4\alpha\sin^2\beta-2\sin^3\alpha\sin\beta\cos\alpha\cos\beta}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{\sin^2\alpha(1-\sin^2\alpha)(1-\sin^2\beta)+\sin^2\beta+2\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta+\sin^4\alpha\sin^2\beta-2\sin^3\alpha\sin\beta\cos\alpha\cos\beta}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{\sin^4\alpha\sin^2\beta+\sin^2\alpha\cos^2\alpha\cos^2\beta+\sin^2\beta-2\sin^3\alpha\sin\beta\cos\alpha\cos\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta+2\sin^2\alpha\sin^2\beta}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\sqrt{\frac{\left(\sin^2\alpha\sin\beta-\sin\alpha\cos\alpha\cos\beta+\sin\beta\right)^2}{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}$$

odnosno

$$\cos\measuredangle ACD=\frac{\sin^2\alpha\sin\beta-\sin\alpha\cos\alpha\cos\beta+\sin\beta}{\sqrt{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}$$

Sledi da je

$$\cos\measuredangle FCH=\cos(\measuredangle ACD-\measuredangle ACH)=\cos\measuredangle ACD\cos\measuredangle ACH+\sin\measuredangle ACD\sin\measuredangle ACH=$$

$$=\frac{\sin^2\alpha\sin\beta-\sin\alpha\cos\alpha\cos\beta+\sin\beta}{\sqrt{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}\sin\alpha+\frac{\sin\alpha\sin(\alpha+\beta)}{\sqrt{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}\cos\alpha=$$

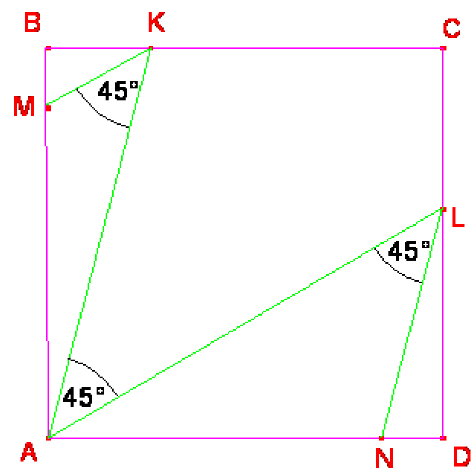
$$=\frac{\sin^3\alpha\sin\beta-\sin^2\alpha\cos\alpha\cos\beta+\sin\alpha\sin\beta+\sin^2\alpha\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos^2\alpha}{\sqrt{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=$$

$$=\frac{\sin^3\alpha\sin\beta+\sin\alpha\sin\beta+\sin\alpha\sin\beta-\sin^3\alpha\sin\beta}{\sqrt{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}=\frac{2\sin\alpha\sin\beta}{\sqrt{2\sin^2\alpha+2\sin^2\beta-\sin^2(\alpha+\beta)}}$$

što nam zajedno sa (6) daje da je zaista $HF\perp CE$. Q.E.D.

2. Na stranicama AB, BC, CD, DA kvadrata $ABCD$ odabrane su tačke M, K, L, N, redom, takve da je $\measuredangle MKA=\measuredangle KAL=\measuredangle ALN=45^o$. Dokazati da je

$$MK^2+AL^2=AK^2+NL^2$$



Rešenje (trigonometrija):

Neka je $AB = a$, $\measuredangle BAK = \varphi$. Iz pravouglog trougla BAK zaključujemo:

$$AK = \frac{a}{\cos\varphi}$$

Posmatrajući trougao AKM, na osnovu sinusne teoreme imamo

$$\frac{MK}{\sin\varphi} = \frac{AK}{\sin(\pi - (\frac{\pi}{4} + \varphi))} \Rightarrow MK = \frac{AK\sin\varphi}{\sin(\frac{\pi}{4} + \varphi)} = \frac{a\sin\varphi}{\cos\varphi\sin(\frac{\pi}{4} + \varphi)}$$

Dalje je $\measuredangle DAL = \dfrac{\pi}{4} - \varphi$, pa iz pravouglog trougla DAL zaključujemo

$$AL = \frac{a}{\cos(\frac{\pi}{4} - \varphi)}$$

Slično, primenom sinusne teoreme na trougao ALN dobijamo

$$\frac{LN}{\sin(\frac{\pi}{4} - \varphi)} = \frac{AL}{\sin(\pi - (\frac{\pi}{4} - \varphi + \frac{\pi}{4}))} \Rightarrow LN = \frac{AL\sin(\frac{\pi}{4} - \varphi)}{\sin(\frac{\pi}{2} + \varphi)} = \frac{a\sin(\frac{\pi}{4} - \varphi)}{\cos(\frac{\pi}{4} - \varphi)\cos\varphi}$$

Sada možemo sastaviti sledeći ekvivalencijski lanac:

$$MK^2 + AL^2 = AK^2 + NL^2 \iff \frac{a^2\sin^2\varphi}{\cos^2\varphi\sin^2(\frac{\pi}{4} + \varphi)} + \frac{a^2}{\cos^2(\frac{\pi}{4} - \varphi)} = \frac{a^2}{\cos^2\varphi} + \frac{a^2\sin^2(\frac{\pi}{4} - \varphi)}{\cos^2\varphi\cos^2(\frac{\pi}{4} - \varphi)} \iff$$

$$\iff \sin^2\varphi\cos^2(\frac{\pi}{4} - \varphi) + \cos^2\varphi\sin^2(\frac{\pi}{4} + \varphi) = \sin^2(\frac{\pi}{4} + \varphi)\cos^2(\frac{\pi}{4} - \varphi) + \sin^2(\frac{\pi}{4} - \varphi)\sin^2(\frac{\pi}{4} + \varphi) \iff$$

$$\iff \sin^2\varphi - \sin^2\varphi\sin^2(\frac{\pi}{4} - \varphi) + \sin^2(\frac{\pi}{4} + \varphi) - \sin^2\varphi\sin^2(\frac{\pi}{4} + \varphi) = \sin^2(\frac{\pi}{4} + \varphi) - \sin^2(\frac{\pi}{4} + \varphi)\sin^2(\frac{\pi}{4} - \varphi) + \sin^2(\frac{\pi}{4} - \varphi)\sin^2(\frac{\pi}{4} + \varphi) \iff$$

$$\iff \sin^2\varphi - \sin^2\varphi\sin^2(\frac{\pi}{4} - \varphi) - \sin^2\varphi\sin^2(\frac{\pi}{4} + \varphi) = 0 \iff \sin^2\varphi\left(1 - \sin^2(\frac{\pi}{4} - \varphi) - \sin^2(\frac{\pi}{4} + \varphi)\right) = 0$$

Imamo da je

$$1 - \sin^2(\frac{\pi}{4} - \varphi) - \sin^2(\frac{\pi}{4} + \varphi) = 1 - \left(\frac{\sqrt{2}}{2}\cos\varphi - \sin\varphi\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\cos\varphi + \sin\varphi\frac{\sqrt{2}}{2}\right)^2 =$$

$$= 1 - \frac{1}{2}\cos^2\varphi + \frac{1}{2}\sin\varphi\cos\varphi - \frac{1}{2}\sin^2\varphi - \frac{1}{2}\cos^2\varphi - \frac{1}{2}\sin\varphi\cos\varphi - \frac{1}{2}\sin^2\varphi = 1 - \cos^2\varphi - \sin^2\varphi = 0$$

Ovim je zadatak rešen.